STABILIZATION OF THE LINEAR SYSTEM OF MAGNETOELASTICITY

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Abstract. We give a necessary and sufficient condition, of geometrical type, for the uniform decay of energy of solutions of the linear system of magnetoelasticity in a bounded domain with smooth boundary. A Dirichlet-type boundary condition is assumed. When the geometrical condition is not fulfilled, we show polynomial decay of the energy, for smooth initial conditions. Our strategy is to use micro-local defect measures to show suitable observability inequalities on high-frequency solutions of the Lamé system.

1. Introduction

1.1. **The system of magnetoelasticity.** Let Ω be a bounded, simply connected domain of $\mathbb{R}^3$, with a smooth boundary. Let us consider the following system, modelling the displacement of a elastic solid in a magnetic field:

$$
\begin{aligned}
&\left.\begin{aligned}
\partial_t^2 v - \mu\Delta v - (\lambda+\mu)\nabla \mathrm{div} v - \kappa \ \mathrm{rot} h \wedge \boldsymbol{B} = 0\\
\beta\partial_t h + \ \mathrm{rot}\ \mathrm{rot} h - \beta \ \mathrm{rot}\ (\partial_t v \wedge \boldsymbol{B}) = 0\\
\mathrm{div}\, h = 0
\end{aligned}\right\} (t,y)\in(0,\infty)\times\Omega \\
&v=0,\quad h.\mathbf{n}=0,\quad \mathrm{curl}\, h\wedge \mathbf{n}=0 \quad (t,y)\in(0,\infty)\times\partial\Omega,
\end{aligned}
\tag{1}
$$

where $v=(v_1,v_2,v_3)$ is the displacement vector of the solid, and $h=(h_1,h_2,h_3)$ the magnetic field. The system is located in a constant exterior magnetic field $\boldsymbol{B}=(B,0,0)$. We have denoted by Δ, ∇, div, curl respectively the Laplace operator, gradient, divergence and curl operators according to the space variable y, in the euclidian metric of $\mathbb{R}^3$. The positive constants κ and β are coupling constants, and $\mathbf{n}$ is the external normal vector to the boundary of Ω. The real Lamé constants λ and μ are such that: $\lambda+2\mu>0$, $\mu>0$ and $\lambda+\mu\neq 0$.

The system (1) has a natural time-decreasing energy:

$$E(t)=\frac{1}{2}\int_\Omega |\partial_t v|^2+\mu|\nabla v|^2+(\lambda+\mu)|\mathrm{div}\, v|^2+\kappa|h|^2 dy.$$

When Ω is simply connected, G. Perla Menzala and E. Zuazua have showed that this energy tends to zero as time tends to infinity, which is a simple consequence, using La Salle invariance principle, of the non-existence of stationnary solution for (1). The goal of this paper is to give estimates on the speed of this convergence.

The system (1) may be seen as a coupling between the Lamé system:

$$\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div}\, u = 0, \tag{2}$$

with Dirichlet boundary conditions, which is a conservative system, and the following heat equation:

$$\beta\partial_t g - \Delta g = 0.$$

The decay of energy is produced by this strongly dissipative equation. From the point of view of v, the dissipation is caused by the coupling term: $R(v) := \mathrm{curl}\,(\partial_t v\wedge \boldsymbol{B})$.

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Let us first consider the uniform decay with respect to initial condition of the energy:

$$E(t) \leq f(t)E(0), \quad f(t) \underset{t\to+\infty}{\longrightarrow} 0, \tag{3}$$

where f if independent of the initial condition. In this case it is easy to show, using the semi-group property of the equation (1), that f maybe taken as a negative exponential function.

In paragraph 1.2 we state, with a technical hypothesis on Ω, a necessary and sufficient condition on the geometry of the problem for (3) to hold. When this condition is not fulfilled, there exist rays on Ω, named $\boldsymbol{B}$-resistant rays, along which the energy of some solutions of (1) concentrates, and the dissipative term $R(v)$ is very small. Indeed, when such a ray exists, there is a sequence of solutions of (1) concentrating on the ray and which is in first approximation parallel to $\boldsymbol{B}$.

When there is no uniform stabilization we show (with the same technical property on Ω than before), that solutions of (1) decay with polynomial speed for smooth enough initial data (cf paragraph 1.3). The speed of decay still depends on the geometry of Ω. In this case, the possible existence of boundary $\boldsymbol{B}$-resistant rays (i.e. living only in the boundary of Ω) of infinite life-length is the main obstacle to the decay.

Before giving more explicit results, let us mention some earlier works on related subjects. As it was already stated, the convergence to 0 for the energy of magnetoelasticity in a bounded, simply connected domain was shown by G. Perla Menzana and E. Zuazua in [11], but their method does not give any information on the rate of convergence. By energy methods, Muñoz Rivera and Racke [12], Muñoz Rivera and de Lima Santos [13] have shown the rate of convergence to be at least polynomial, in dimension 2 or 3, but only for some precise types of domains. Andreou and Dassios [1] have examined the same system on the entire space $\mathbb{R}^3$, showing again polynomial decay for some initial conditions.

The linear system of thermoelasticity has been more precisely understood. In this system, the Lamé equations are coupled with a scalar heat equation. The dissipation is caused by the longitudinal part of the Lamé equation (the curl-free part of v). In [9] and [2], the authors give (under a spectral assumption) a necessary and sufficient condition on Ω, of geometrical nature, for the uniform decay in dimension 2 or 3. Namely, this decay is equivalent to the non-existence of rays, called "transversal polarization rays", carrying the transversal component of v (the divergence-free component), which resists to the dissipation. In [9], the authors also prove the polynomial decay in dimension 2, under the same spectral assumption, which is namely that the operator associated to the equation does not admit any real eigenvalue. As shown in [11], this spectral condition is always fulfilled for the system of magnetoelasticity in a bounded, simply-connected domain.

The comparison of the two systems of thermo and magnetoelasticity show that thermoelasticity is slightly less dissipative (the coupling of the Lamé system with the heat equation is weaker), and more difficult to describe, because of the non-trivial polarization of transversal waves.

1.2. **Uniform decay.** Assume that $\partial\Omega$ has no contact of infinite order with its tangents. Thus, the hamiltonian flow of the symbol of a d'Alembertian $\partial_t^2 - c^2\Delta$, which is defined locally in $S^*(\mathbb{R}\times\Omega)$ (the spherical cotangent bundle of Ω), maybe extended until the boundary of this bundle to a global flow, the generalized bicharacteristic flow, wich may be seen as a continuous flow on the spherical compressed cotangent bundle $S_b^*(\mathbb{R}\times\overline{\Omega})$ (cf [7, chap. 24.3]). We shall call bicharacteristic rays or just rays the characteristic curves of this flow. Such a curve γ will be said parallel to $\boldsymbol{B}$ if its direction of propagation is always parallel to $\boldsymbol{B}$ and orthogonal to $\boldsymbol{B}$ if its direction of propagation is always orthogonal to $\boldsymbol{B}$. We refer to section 3 for the exact definitions of $S_b^*(\mathbb{R}\times\overline{\Omega})$ and of the generalized bicharacteristic flow.

The Lamé system (2) may be written as the sum of two wave equations known as the longitudinal and transversal wave equations, of respective speed $c_L := \sqrt{\lambda+2\mu}$ and $c_T := \sqrt{\mu}$ (cf paragraph 3.5). The assumption $\lambda+\mu\neq 0$ is equivalent to $c_L \neq c_T$.

Definition 1.1. One calls **longitudinal ray** (respectively **transversal ray**) any bicharacteristic ray for the operator $\partial_t^2 - c_L^2\Delta$ (respectively $\partial_t^2 - c_T^2\Delta$). One calls $\boldsymbol{B}$**-resistant ray** any **continuous** application

γ from an open interval $I = (s_0, s_n)$ to $S_b^*(\mathbb{R} \times \overline{\Omega})$ such that there exists a finite number of reals $s_0 < s_1 < ... < s_n$ such that:

- on (s_{j-1}, s_j), $j \in \{1, ..., n\}$, γ is a longitudinal ray parallel to $\boldsymbol{B}$, or a transversal ray orthogonal to $\boldsymbol{B}$;
- if $j \in \{1, ..., n-1\}$, $\gamma(s_j)$ is an hyperbolic point for the longitudinal and transversal waves (cf paragraph 3.1.7) and one of the following assertions is true:
  - $(L \to T)$ case: γ is a longitudinal ray on $]s_{j-1}, s_j[$, and a transversal ray on $]s_j, s_{j+1}[$;
  - $(T \to L)$ case: γ is a transversal ray on $]s_{j-1}, s_j[$, and a longitudinal ray on $]s_j, s_{j+1}[$.

(cf figure 1)

Near s_j, $1 \leq j \leq n-1$, the continuity imposed by the definition of γ gives a condition on the angles of incidence and refraction. In the case $(L \to T)$, if we denote by α_L the angle between the longitudinal incoming ray and the tangent to $\partial\Omega$ in the plane of incidence, and by β_T the angle between the transversal outcoming ray and this tangent (cf figure 1, c), we have:

$$\tan \alpha_L = \frac{c_T}{c_L}, \quad \tan \beta_T = \frac{c_L}{c_T},$$

(which implies $\alpha_L + \beta_T = \pi/2$). In the case $(T \to L)$, and with similar notations, we have:

$$\tan \alpha_T = \frac{c_L}{c_T}, \quad \tan \beta_L = \frac{c_T}{c_L}.$$

Remark 1.2. The $\boldsymbol{B}$-resistant rays of figure 1 are all planar, but this is not a general property.

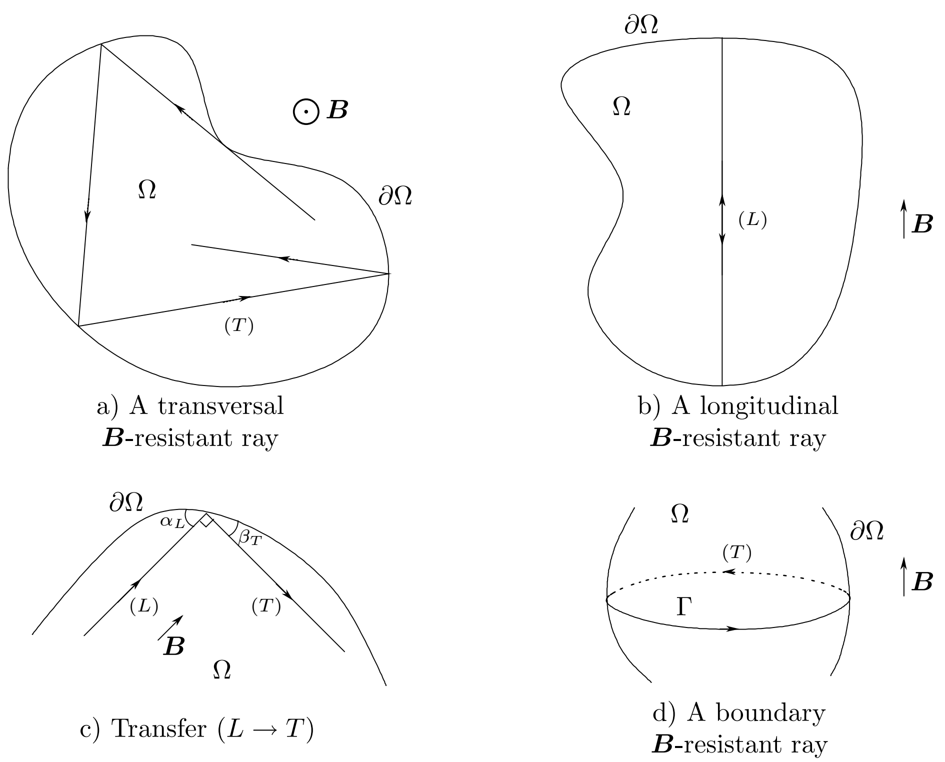

FIGURE 1. Examples of $\boldsymbol{B}$-resistant rays

Theorem 1. *Let Ω be a bounded, simply connected domain of $\mathbb{R}^3$, with a smooth boundary, having no contact of infinite order with its tangents.*
The energy of the system of magnetoelasticity in Ω decays uniformly if and only if there exists an $L > 0$ such that every $\boldsymbol{B}$-resistant ray on Ω is of length at most L.

Remark 1.3. As it will be shown in the proof, the transversal rays carry the component of v which is orthogonal to the direction of propagation, and the longitudinal rays the component of v which is parallel to this direction. A $\boldsymbol{B}$-resistant ray, whose direction of propagation is orthogonal to $\boldsymbol{B}$ in the transversal case and parallel to $\boldsymbol{B}$ in the longitudinal case carries essentially the component of v which is parallel to $\boldsymbol{B}$, thus cancelling the dissipative term:

$$R(v) := \operatorname{curl}(\partial_t v \wedge \boldsymbol{B}).$$

From this point of view, the theorem 1 is very natural.

Remark 1.4. It is essential to assume $c_L \neq c_T$. Otherwise, the first equation in (1) would be a wave equation with wave speed $c_l = c_T$. Every solution of (1) such that:

$$v_{\restriction t=0} \perp \boldsymbol{B}, \quad \partial_t v_{\restriction t=0} \perp \boldsymbol{B}, \quad h_{\restriction t=0} = 0$$

would be of constant energy.

Remark 1.5. If Ω is not simply connected, there exists a finite dimensionnal space E of stationary solutions of (1), whose components along v are null. The study of the decay to zero of the solutions may be replaced by the study of their convergence to the eigenfunctions corresponding to the space E (cf [11, chap. 5]). We won't develop this aspect here.

Remark 1.6. The condition of uniform decay is not fulfilled in simple cases, like the one of a bowl, but is generic in the class of C^∞ open sets.

1.3. **Polynomial decay.** Now we state a result of polynomial decay for initial data which are sufficiently smooth. The existence of a boundary $\boldsymbol{B}$-resistant ray of infinite life-length is equivalent to the existence of a smooth closed curve Γ of $\partial\Omega$, included in a plane $\mathcal{P}$ normal to $\boldsymbol{B}$, boundary of a convex set of $\mathcal{P}$, and such that on Γ, $\mathbf{n}$ is normal to $\boldsymbol{B}$ (cf figure 1, d). On such a curve, $\boldsymbol{B}$ stays tangential to the boundary. Let $\aleph(\Gamma)$ be the minimal order of contact of $\partial\Omega$ with a tangent parallel to $\boldsymbol{B}$. If such a curve Γ exists, and if the boundary has no contact of infinite order with its tangents, then:

$$2 \leq \aleph(\Gamma) < \infty.$$

If (v, h) is a sufficiently smooth solution of (1), we shall denote by $E^{(j)}(t)$ the energy of order j of (v, h):

$$E^{(j)}(t) := \frac{d^j}{dt^j} E(t).$$

Let X_j be the subspace of X of all initial data of (1) such that $E^{(j)}(0)$ is finite. It is exactly the domain of $\mathcal{A}^j$, where $\mathcal{A}$ is the linear operator of magnetoelasticity defined in section 2.

Theorem 2. *Let Ω be a bounded, simply connected domain with smooth boundary having no contact of infinite order with its tangents.*
a) *Assume there is no boundary $\boldsymbol{B}$-resistant ray on Ω of infinite life-length. Then:*

$$\exists C > 0, \quad \forall V_0 \in X_1, \quad \forall t \geq 0, \quad E(t) \leq \frac{C}{t+1} E^{(1)}(0).$$

b) *Assume on the contrary that such rays exist. Le $\Gamma_1, ..., \Gamma_M$ be the support of this rays, and:*

$$K := \sup_{m=1..M} \aleph(\Gamma_m).$$

Then:

$$\exists C > 0, \quad \forall V_0 \in X_K, \quad \forall t \geq 0, \quad E(t) \leq \frac{C}{t+1} E^{(K)}(0).$$

Remark 1.7. By an easy interpolation argument, one may deduce from **b** the polynomial decay of any solution with initial condition in X_1:

$$\forall V_0 \in X_1, \quad \forall t \geq 0, \quad E(t) \leq \frac{C}{(t+1)^{\frac{1}{K}}} E^{(1)}(0).$$

Remark 1.8. Theorem 2 completes the works of J. E. Muñoz Rivera et M. De Lima Santos [13] which show, for some types of domains of $\mathbb{R}^3$, a decay in $1/t$ for initial data in $E^{(7)}$. Note that the domains considered in their work (all of which have contacts of infinite order with their tangents) do not fall within the scope of our article.

The remainder of the paper is organized as follows. In section 2, we reduce theorems 1 and 2 to high-frequency observability inequalities on the Lamé system (2). This is based on two arguments: the setting aside of low frequencies, which is a consequence of the non-existence of stationnary solution for the equation (1) shown in [11], and the decoupling, by simple calculations, of the two equations (the Lamé system and the heat equation) which compose (1). In section 3, we introduce micro-local defect measures (an object due to P. Gérard [6] and L. Tatar [16], and in this particular setting to N. Burq and G. Lebeau [2]), in order to study the lack of compactness of a sequence of high-frequency solutions of the Lamé system. The main result of this section (apart from the existence of the measures), is a propagation theorem which was stated and shown in [2]. In section 4, we prove the observability inequality on solutions of the Lamé system (2) which implies theorem 1. The method of proof is to introduce, in a contradiction argument, a sequence of high frequency solutions of (2) which contredicts this inequality, and to use propagation arguments on the defect measures of this sequence. Section 5 is devoted to the necessary condition of theorem 1, and is inspired by [5]: defect measures are used to construct a sequence of solutions of (2) concentrating on a $\boldsymbol{B}$-resistant ray and contradicting an observability inequality. Finally, in section 6, we prove by similar arguments than those of section 4 an observability inequality with loss of derivatives which implies the polynomial decay.

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2. Observability inequality for the Lamé system

2.1. **Notations and preliminary results.** In this subsection are gathered a few basic facts about equations (1) and (2), as well as some notations. The main results of section 2 are stated in the next subsection.

If U is an open set of $\mathbb{R}^3$ or $\mathbb{R}^4$ we set:

$$\boldsymbol{H}^s(U) := H^s(U, \mathbb{C}^3), \quad \boldsymbol{L}^2(U) := L^2(U, \mathbb{C}^3).$$

2.1.1. *Magnetoelasticity.* Consider the following spaces:

$$H := \left\{ g \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} g = 0 \text{ in } \Omega,\ g.n = 0 \text{ in } \partial\Omega \right\}$$
$$\boldsymbol{H}_0^1 := \{ f \in \boldsymbol{H}^1(\Omega),\ f = 0 \text{ in } \partial\Omega \}$$
$$H' := \left\{ f \in H \cap \boldsymbol{H}^2(\Omega),\ \operatorname{curl} f \wedge n = 0 \text{ in } \partial\Omega \right\},$$

and the following norms:

$$\|g\|_H^2 := \kappa \|g\|^2_{\boldsymbol{L}^2(\Omega)}, \quad \|f\|^2_{\boldsymbol{H}_0^1(\Omega)} := (\lambda + \mu) \|\operatorname{div} f\|^2_{L^2(\Omega)} + \mu \|\nabla f\|^2_{(\boldsymbol{L}^2(\Omega))^3}.$$

Let $\mathcal{A}$ be the unbounded operator on $X := \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \times H$, with domain $D(\mathcal{A})$, defined by:

$$\mathcal{A}(V_0) := \begin{pmatrix} -v_1 \\ -\Delta_e v_0 - \kappa (\operatorname{curl} h_0) \wedge \boldsymbol{B} \\ -\operatorname{curl}(v_1 \wedge \boldsymbol{B}) + \frac{1}{\beta} \operatorname{curl} \operatorname{curl} h_0 \end{pmatrix}$$
$$D(\mathcal{A}) := (\boldsymbol{H}^2 \cap \boldsymbol{H}_0^1) \times \boldsymbol{H}_0^1 \times H'.$$

where $V_0=(v_0,v_1,h_0)$ denotes an element of X. Equation (1) may be rewritten:

$$\partial_t V+\mathcal{A}V=0,\quad V=(v,\partial_t v,h). \tag{4}$$

The following proposition is due to G. Perla Menzala and E. Zuazua [11]:

Proposition 2.1. *a) The operator $\mathcal{A}$ is maximal accretive. For any initial data $V_0\in X$, there exists an unique weak solution $V(t)=(v(t),\partial_t v(t),h(t))\in C^0([0,+\infty[;X)$ of* (4) *such that $V(0)=V_0$. Functions v and h are solutions in the distributional sense of the three first lines of system* (1)*.*
b) The energy $E(t)$:

$$E(t):=\frac{1}{2}\|V(t)\|_X^2=\frac{1}{2}\int_\Omega |\partial_t v|^2+\mu|\nabla v|^2+(\lambda+\mu)|\operatorname{div} v|^2+\kappa|h|^2dy \tag{5}$$

is decreasing. More precisely:

$$\forall t\geq 0,\quad E(t)-E(0)=-\frac{\kappa}{\beta}\int_0^t\int_\Omega |roth|^2dy. \tag{6}$$

c) If Ω is simply connected:

$$\forall V_0\in X,\quad E(t)\underset{t\to+\infty}{\longrightarrow}0.$$

The assertions a) and b) are straightforward applications of the semi-group theory for the oprator $\mathcal{A}$. The assertion c) is a consequence of the non-existence of stationnary solutions for the system.

2.1.2. *Lamé system.* Let us now consider the Lamé system with Dirichlet boundary conditions:

$$\begin{gathered}\partial_t^2u-\Delta_e u=0 \text{ in } \mathbb{R}\times\Omega\\ u_{\restriction\partial\Omega}=0\\ (u_{\restriction t=0},\partial_t u_{\restriction t=0})=(u_0,u_1).\end{gathered} \tag{7}$$

Let X_e be the space $\boldsymbol{H}_0^1\times\boldsymbol{L}^2$ and $\mathcal{L}$ the unbounded operator on X_e defined by:

$$\mathcal{L}:=\begin{bmatrix}0 & -\mathrm{Id}\\ -\Delta_e & 0\end{bmatrix}\qquad D(\mathcal{L}):=\boldsymbol{H}^2\cap\boldsymbol{H}_0^1\times\boldsymbol{L}^2. \tag{8}$$

Taking (u_0,u_1) in the energy space X_e, the equation (7) may be written:

$$\partial_t U+\mathcal{L}U=0,\ U(t)=(u,\partial_t u). \tag{9}$$

Proposition 2.2. *The operator $\mathcal{L}$ is maximal and unitary. For any initial data $U_0=(u_0,u_1)\in X_e$, the system (9) has an unique weak solution $U\in C^0(\mathbb{R},X_e)$. Furthermore, the function u is a solution of (7) in the distributional sense. At last, the energy:*

$$\frac{1}{2}\left(\|u(t)\|^2_{\boldsymbol{H}^1}+\|\partial_t u\|^2_{\boldsymbol{L}^2}\right)=\frac{1}{2}\|U\|^2_{X_e}$$

of this solution is constant.

2.1.3. *Two useful lemma.* The two following standard lemma will be of great help in all this paper. The first one is due to the fact that Ω is simply connected (cf [17, Appendix I, lemma 1.6]):

Lemma 2.3. *The $\boldsymbol{H}^1$ norm on $H\cap\boldsymbol{H}^1$ is equivalent to the norm: $\|u\|:=\|\operatorname{curl} u\|_{\boldsymbol{L}^2}$.*

The second lemma is a elementary energy estimate on solutions of the non-homogeneous Lamé system. If $w(t)$ is a function with values in some Hilbert space, we set: $(w_0,w_1):=(w,\partial_t w)_{\restriction t=0}$.

Lemma 2.4. *Let $T > 0$, $W \in C^0((0,T), X_e)$ and $F \in L^2((0,T), X_e)$ such that:*

$$\partial_t W + \mathcal{L}W = F, \quad t \in (0,T). \tag{10}$$

Then:

$$\int_0^T \|W(t)\|_{X_e}^2 \, dt \le C \left\{ \|W(0)\|_{X_e}^2 + \int_0^T \|F(t)\|_{X_e}^2 dt \right\},$$

where C only depends on T. In particular, if:

$$w \in C^1((0,T), \boldsymbol{L}^2(\Omega)) \cap C^0((0,T), \boldsymbol{H}_0^1(\Omega)), \quad f \in \boldsymbol{L}^2((0,T)\times\Omega)$$
$$\partial_t^2 w - \Delta_e w = f.$$

Then:

$$\|w\|_{\boldsymbol{H}^1((0,T)\times\Omega)}^2 \le C \left(\|w_0\|_{\boldsymbol{H}^1(\Omega)}^2 + \|w_1\|_{\boldsymbol{L}^2(\Omega)}^2 + \|f\|_{\boldsymbol{L}^2((0,T)\times\Omega)}^2 \right).$$

Proof. To prove the first inequality, we may suppose $W_0 \in D(\mathcal{L})$. The X_e scalar product of (10) with W gives

$$\begin{aligned}
\mathrm{Re}(\partial_t W, W)_{X_e} + \underbrace{\mathrm{Re}(\mathcal{L}W, W)_{X_e}}_{0} &= (W,F)_{X_e} \\
\frac{1}{2}\frac{d}{dt}\|W(t)\|_{X_e}^2 &\le \|W(t)\|_{X_e} \|F(t)\|_{X_e} \\
\frac{d}{dt}\|W(t)\|_{X_e} &\le \|F(t)\|_{X_e} \\
\|W(s)\|_{X_e} &\le \|W_0\|_{X_e} + \int_0^s \|F(t)\|_{X_e} \, dt, \quad s \in (0,T) \\
\|W(s)\|_{X_e}^2 &\le C \left\{ \|W_0\|_{X_e}^2 + s \int_0^s \|F(t)\|_{X_e}^2 \, dt \right\}.
\end{aligned}$$

(the last line is a consequence of Cauchy-Schwarz inequality). Next, we bound, in the right member of the inequality the integral from 0 to s by the same integral from 0 to T, and we integrate with respect to s between 0 and T, which yields the first part of the lemma. The second part is an easy consequence of it. ☐

2.2. **Results.**

Proposition 2.5 (Uniform decay)**.** *Let Ω be a smooth, simply connected, bounded domain of $\mathbb{R}^3$.*

- **a)** *Assume that there exist $T > 0$ and $C > 0$ such that for any solution U of (9):*

$$\|u_0\|_{\boldsymbol{H}_0^1}^2 + \|u_1\|_{\boldsymbol{L}^2}^2 \le C \left(\|\mathrm{curl}\,(\partial_t u \wedge \boldsymbol{B})\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2 + \|u_0\|_{\boldsymbol{L}^2}^2 + \|u_1\|_{\boldsymbol{H}^{-1}}^2 \right) \tag{11}$$

  then the energy of solutions of the system of magnetoelasticity (1) decays uniformly with respect to initial data.
- **b)** *Conversely, if the energy of solutions of (1) decays uniformly, then there exist $T > 0$ and $C > 0$ such that for any solution of (7) of finite energy:*

$$\|u_0\|_{\boldsymbol{H}_0^2}^2 + \|u_1\|_{\boldsymbol{L}^2}^2 \le C \int_0^T \|\partial_t u \wedge \boldsymbol{B}\|_{\boldsymbol{L}^2(\Omega)}^2. \tag{12}$$

Remark 2.6. The two inequalities (11) and (12) are indeed equivalent (by theorem 1).

Let $U = (u, \partial_t u)$ be a solution to (9) with initial data $U_0 \in D(\mathcal{L}^N)$. Set:

$$\mathcal{Q}_T^N(u) := \sum_{l=0}^N \|\mathrm{curl}\,(\partial_t^{l+1} u \wedge B)\|_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}^2.$$

Proposition 2.7 (A sufficient condition of polynomial decay)**.** *Let Ω be a bounded, simply connected domain of $\mathbb{R}^3$. Assume that there exist $T > 0$, $C > 0$ and an integer $N \geq 1$ such that for every solution of the Lamé system* (9) *with initial data:*

$$U_0 = (u_0, u_1) \in D(\mathcal{L}^N), \tag{13}$$

the following inequality holds:

$$\|u_0\|^2_{\boldsymbol{H}^1_0} + \|u_1\|^2_{\boldsymbol{L}^2} \leq C \left(\mathcal{Q}^N_T(u) + \|u_0\|^2_{\boldsymbol{L}^2} + \|u_1\|^2_{\boldsymbol{H}^{-1}} \right). \tag{14}$$

Then there exists $C > 0$ such that for every solution V of (4) *with initial date $V_0 \in D(\mathcal{A}^N)$, and for all positive t,*

$$\|V(t)\|^2_X \leq \frac{C}{t+1} \|V_0\|^2_{D(\mathcal{A}^N)}$$

Remark 2.8. results such as propositions 2.5 and 2.7 are fairly classical in this setting. To prove them, we shall avoid the usual abstract decoupling argument (see [9]) but rather use simple energy estimates on systems of magnetoelasticity and Lamé.

2.3. **Uniform decay.** We prove here the proposition 2.5. We first write a necessary and sufficient condition of uniform decay for solutions of a general dissipative equation. The second step of the proof consists in applying this condition to the system of magnetoelasticity, furthermor decoupling it in the system of Lamé and an heat equation.

2.3.1. *Abstract framework.* Let $\mathcal{P}$ be a maximal, accretive operator on an Hilbert space X, with dense domain $D(\mathcal{P})$. Denote by $\|..\|$ the norm of X, $\|..\|_1$ the natural norm of $D(\mathcal{P}^1)$ and $\|..\|_{-1}$ the norm of its dual space, with respect to the pivot space X. Assume the embedding:

$$X \longrightarrow D(\mathcal{P})'$$

is compact. For $z_0 \in X$, we will denote by $z(t)$ the solution (obtained for example by standard semi-group theory) of:

$$\frac{d\,z}{dt} + \mathcal{P}z = 0, \quad z_{t=0} = z_0 \tag{15}$$

By accretivity of $\mathcal{P}$, the energy $\frac{1}{2}\|z\|^2$ is time-decreasing. The following uniqueness-compactness argument is by now classical (cf [3]):

Lemma 2.9. *The two following assertions (i) and (ii) are equivalent:*

(i) $\exists C > 0,\ \exists a > 0,\ \forall z_0 \in X,\ \forall t > 0,\ \|z(t)\|^2 \leq C\|z_0\|^2 e^{-at}$

(the energy is uniformly decreasing)

(ii) *a)* $\exists T > 0,\ \exists C > 0,\ \forall z_0 \in X,\ \|z(T)\|^2 \leq C \left(\|z(0)\|^2 - \|z(T)\|^2 + \|z(0)\|^2_{-1} \right)$

b) There is no non-zero solution of (15) of constant energy on $[0, +\infty[$.

Corollary 2.10. *The energy of (4) is uniformly time-decreasing if and only if:*

$$\exists T > 0,\ \exists C > 0,\ \forall V_0 \in X,\ E(T) \leq C \left\{ \int_0^T \int_\Omega |\operatorname{curl} h|^2 \, dy \, dt + \|V_0\|^2_{D(\mathcal{A})'} \right\}.$$

Indeed, the non-existence of stationnary solution (the condition ii,b of lemma 2.9 has been proved in [11, p.356]), which shows the corollary.

Proof of lemma 2.9. It is easy to see that (i) may be replaced by:

(i') $\exists T > 0,\ \exists C > 0,\ \forall z_0 \in X, \quad \|z(T)\|^2 \leq C \left(\|z(0)\|^2 - \|z(T)\|^2 \right)$

Clearly (i') implies (ii).

Assume (ii). For some $T > 0$, set:

$$q_T(z) := \|z(0)\|^2 - \|z(T)\|^2, \quad G^T := \{z_0 \in X,\ q_T(z) = 0\},$$

which is the kernel of a positive, bounded, quadratic form on X, thus a closed subspace of X.

According to (ii), a), and the compactness of the embedding from X to $D(P)'$, $G(T)$ is locally compact thus of finite dimension, for large T. By assumption b),

$$\bigcap_{T \geq 0} G^T = \{0\}.$$

Consequently, $\dim G^T$ being a time decreasing function of T, when T is large enough:

$$G^T = \{0\} \tag{16}$$

Let's fix such a T. The quadratic form q_T is positive definite so that its square root $\sqrt{q_T}$ is a pre-hilbertian norm on X, bounded from above by the natural norm of X. Assume (i') does not hold. Then there exists a sequence (z_0^k) of elements of X such that:

$$1 = \|z^k(T)\|^2, \qquad \lim_{k \to +\infty} q_T(z^k) = 0. \tag{17}$$

This implies that $\|z_0^k\|$ is bounded. Thus, we may extract form (z_0^k) a subsequence, which we will again denote by (z_0^k), such that:

$$z_0^k \underset{k \to +\infty}{\rightharpoonup} z_0 \in X, \text{ weakly in } X.$$

Let φ_T be the hermitian product given by q_T. We have:

$$\lim_{k \to +\infty} \varphi_T(z^k, z) \underset{k \to +\infty}{\longrightarrow} q_T(z),$$

which implies, with (17), that $q_T(z) = 0$ and thus, using (16) that $z = 0$. The compactness of the embedding of X in $D(P)'$ yields:

$$\lim_{k \to +\infty} \|z_0^k\|_{-1} = 0.$$

Using a) and (17) we obtain the following contradictory assertion:

$$1 \leq C\left(q_T(z^k) + \|z_0^k\|_{-1}^2\right) = o(1) \text{ quand } k \to +\infty.$$

□

2.3.2. *Proof of proposition 2.5.* Assume the uniform time-decay of the energy of solutions of (4). Then, by (6), there exist $T > 0$ and $C > 0$ such that the following estimates hold for any solution v of (4):

$$\|v_0\|_X^2 \leq C \int_0^T \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2(\Omega)}\, dt. \tag{18}$$

Let U be a solution of the Lamé system with initial data $U_0 = (u_0, u_1) \in D(\mathcal{L})$ and V the solution of the system of magnetoelasticity with initial data:

$$V_0 = (v, \partial_t v, h)_{\restriction t=0} = (u_0, u_1, 0).$$

Set: $W(t) := V(t) - \big(u(t), \partial_t u(t), 0\big)$. Then:

$$\partial_t W + \mathcal{A} W = \big(0, 0, -\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\big)$$

Take the scalar product in X with W of the two side of this equality, then inegrate the real part with respect to time between 0 and T. Using:

$$\operatorname{Re}(\mathcal{A}W, W)_X = \frac{\kappa}{\beta}\|\operatorname{curl} h\|^2_{\boldsymbol{L}^2(\Omega)}$$

$$\big(\operatorname{curl}(\partial_t u \wedge \boldsymbol{B}), h\big)_{\boldsymbol{L}^2(\Omega)} = \left(\partial_t u \wedge \boldsymbol{B}, \operatorname{curl} h\right)_{\boldsymbol{L}^2(\Omega)},$$

the fact that $W_{\restriction t=0} = 0$, and in the second line, the inequality (18), we get:

$$\|W(T)\|_X^2 + \int_0^T \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2(\Omega)}\, dt \leq C \int_0^T \|\partial_t u(t) \wedge \boldsymbol{B}\|^2_{\boldsymbol{L}^2(\Omega)}\, dt$$

$$\|u_0\|^2_{\boldsymbol{H}^1_0(\Omega)} + \|u_1\|^2_{\boldsymbol{L}^2(\Omega)} \leq C \int_0^T \|\partial_t u(t) \wedge \boldsymbol{B}\|^2_{\boldsymbol{L}^2(\Omega)}\, dt$$

This shows point b). To prove a), assume that inequality (11) holds. Consider a solution $V = (v, \partial_t v, h)$ of (4) with initial data $V_0 = (v_0, v_1, h_0)$, and the solution u of Lamé system with initial data:

$$(u, \partial_t u)_{\restriction t=0} = (v_0, v_1).$$

Thus, by (11):

$$\|v_0\|^2_{\boldsymbol{H}^1_0} + \|v_1\|^2_{\boldsymbol{L}^2} \leq C \left(\|\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\|^2_{\boldsymbol{H}^{-1}((0,T)\times\Omega)} + \|v_0\|^2_{\boldsymbol{L}^2} + \|v_1\|^2_{\boldsymbol{H}^{-1}} \right). \tag{19}$$

Furthermore, the energy inequality on the non-homogeneous Lamé system (lemma 2.4) yields:

$$\|u - v\|^2_{\boldsymbol{H}^1((0,T)\times\Omega)} \leq C \int_0^T \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2}\, dt$$

$$\|\operatorname{curl}(\partial_t u \wedge \boldsymbol{B})\|^2_{\boldsymbol{H}^{-1}((0,T)\times\Omega)} \leq C \left\{ \int_0^T \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2}\, dt + \|\operatorname{curl}(\partial_t v \wedge \boldsymbol{B})\|^2_{\boldsymbol{H}^{-1}((0,T)\times\Omega)} \right\},$$

which implies, using (19), lemma 2.3) and the following equation:

$$\beta \partial_t h + \operatorname{curl}\operatorname{curl} h = \beta \text{ rot } (\partial_t v \wedge \boldsymbol{B}),$$

$$\|v_0\|^2_{\boldsymbol{H}^1_0} + \|v_1\|^2_{\boldsymbol{L}^2} \leq C \left\{ \int_0^T \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2} dt + \|v_0\|^2_{\boldsymbol{L}^2} + \|v_1\|^2_{\boldsymbol{H}^{-1}} \right\}. \tag{20}$$

In order to use corollary 2.10, we need to add to the left side of inequality (20) the $\boldsymbol{L}^2$-norm of $h(T)$. We may do so by taking a larger T. Indeed, consider $s \in [0, T]$ such that:

$$\|h(s)\|^2_{\boldsymbol{L}^2} = \min_{t\in[0,T]} \|h(t)\|^2_{\boldsymbol{L}^2}.$$

Lemma 2.3 gives an $\alpha > 0$ such that:

$$g \in H \cap \boldsymbol{H}^1 \Rightarrow \|g\|^2_{\boldsymbol{L}^2(\Omega)} \leq \alpha \|\operatorname{curl} g\|^2_{\boldsymbol{L}^2(\Omega)}$$

$$\begin{aligned} \|h(s)\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \frac{1}{T} \int_0^T \|h(t)\|^2_{\boldsymbol{L}^2(\Omega)} dt \\ &\leq \frac{\alpha}{T} \int_0^T \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2(\Omega)} dt. \end{aligned}$$

Inequality (20) taken with initial time $t = s$ yields:

$$\begin{aligned} E(s) = \frac{1}{2} &\left(\|u(s)\|^2_{\boldsymbol{H}^1_0(\Omega)} + \|\partial_t u(s)\|^2_{\boldsymbol{L}^2(\Omega)} + \kappa \|h(s)\|^2_{\boldsymbol{L}^2(\Omega)} \right) \\ &\leq C \left\{ \int_0^{2T} \|\operatorname{curl} h(t)\|^2_{\boldsymbol{L}^2(\Omega)} dt + \|u(s)\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t u(s)\|^2_{\boldsymbol{H}^{-1}(\Omega)} \right\}. \end{aligned}$$

The energy E being time-decreasing, this implies the inequality of corollary 2.10, and so the uniform decay of solutions of (1). The proof of b) is complete.

2.4. **Polynomial decay.**

2.4.1. *Abstract framework.* We shall use here the notations of paragraph 2.3.1 Let N be a positive integer, and Q_T the quadratic form defined by:

$$Q_T(z) := \sum_{l=0}^{N} \left(\|\partial_t^l z(0)\|^2 - \|\partial_t^l z(T)\|^2\right) = \sum_{l=0}^{N} q_T(\partial_t^l z), \quad D(Q_T) = D(\mathcal{P}^N).$$

The function $\partial_t^l z$ being a solution of (15), its energy decays with time, so that Q_T is positive. Recall the definition: (cf [14])

Definition 2.11. The quadratic form Q_T is said to be **closable** when the closure X_Q^T of $D(Q_T)$ in X for the norm:

$$\|z_0\|_{Q_T} = \sqrt{\|z_0\|^2 + Q_T(z)}$$

is complete for this norm.

Remark 2.12. This is equivalent to the fact that for all Cauchy sequence in $D(Q_T)$ for the norm $\|.\|_{Q_T}$, (z^k), converging to 0 in X, we have:

$$\lim_{k \to +\infty} Q_T(z^k) = 0.$$

We shall again assume the compactness of the embedding: $X \longrightarrow D(\mathcal{P})'$..

The following classical argument goes back to Russel [15].

Lemma 2.13. *Under the following assumptions:*

a) *there exist $T, C > 0$ such that:*

$$\forall z_0 \in D(\mathcal{P}^N), \quad \|z(T)\|^2 \le C \left(\|z_0\|_{-1}^2 + Q_T(z)\right); \tag{21}$$

b) *system (15) have no non-zero solution of constant energy on $[0, +infty[$;*

c) *the quadratic form Q_T is closable.*

There exists $C > 0$ such that:

$$\forall z_0 \in D(\mathcal{P}^N),\ \forall t \ge 0, \quad \|z(t)\|^2 \le \frac{C}{t+1} \|z_0\|_N^2. \tag{22}$$

Proof. We shall first use a compactness argument similar to that of proposition 2.9. Let T be a large positive real number, such that (21) holds. Consider X_Q^T, the subspace of X introduced in definition (2.11). Extending Q_T to X_Q^T by continuity, we can still write inequality (21) for $v_0 \in X_Q^T$. Consider the following closed subspace of X_Q^T:

$$J^T := \{z_0 \in X_Q^T,\ Q_T(z) = 0\}.$$

By (21) we have, for any $z_0 \in J^T$:

$$\|z_0\|_{Q_T} \le C \|z_0\|_{-1}.$$

Using assumption *b*) as in the proof of proposition 2.9, we obtain that for T large enough:

$$J^T = \{0\}. \tag{23}$$

From now on, T will be taken such that (23) holds. The same process as in the proof of proposition 2.9 yields:

$$\forall z_0 \in X_Q^T, \qquad \|z(T)\|^2 \le C\, Q_T(z). \tag{24}$$

Elsewhere, there would exist a sequence (z_0^k) of elements X_Q^T such that:

$$\|z^k(T)\|^2 = 1, \quad \lim_{k \to +\infty} Q_T(z^k) = 0. \tag{25}$$

Up to a subsequence, we may assume that (z_0^k) converges weakly to 0 in X_Q^T. By (23) and (25), $z_0 = 0$. By compactness:

$$\|z_0^k\|_{-1} \underset{k \to +\infty}{\longrightarrow} 0.$$

In view of (25), this contredicts (21).

By the triangle inequality, we have:

$$\sqrt{q_T(x+y)} \leq \sqrt{q_T(x)} + \sqrt{q_T(y)}.$$

Noting that $(\mathrm{Id}+\mathcal{P})^l$ is an isomorphism from $D\left(\mathcal{P}^l\right)$ to X, it is easy to show:

$$\begin{aligned} Q_T(z) &\leq C\sum_{l=0}^{N} q_T\left((\mathrm{Id}-\partial_t)^l z\right) \\ &\leq C\sum_{l=0}^{N}\left(\|z(0)\|_l^2 - \|z(T)\|_l^2\right). \end{aligned}$$

From this and (24), we deduce:

$$\begin{aligned} z_0 \in D(\mathcal{P}^N) &\Rightarrow \|z(T)\|^2 \leq C\sum_{l=0}^{N}(\|z(0)\|_l^2 - \|z(T)\|_l^2) \\ z_0 \in X &\Rightarrow \|z(T)\|_{-N}^2 \leq C\sum_{l=0}^{N}(\|z(0)\|_{l-N}^2 - \|z(T)\|_{l-N}^2) \end{aligned}$$

$$z_0 \in X \Rightarrow \|z(T)\|_0^4 \leq C\|z(T)\|_N^2 \sum_{l=0}^{N}\left(\|z(0)\|_{l-N}^2 - \|z(T)\|_{l-N}^2\right) \tag{26}$$

The last inequality follows from the trivial bound: $\|f\|^2 \leq \|f\|_N\|f\|_{-N}$.
Set: $\beta_n := \sum_{j=0}^{N}\|z(nT)\|_{-j}^2$. We have:

$$\beta_n \leq (N+1)\|z(nT)\|^2.$$

Thus, using (26) with the initial data $t = nT$ instead of $t = 0$, there exists a constant $C > 0$ such that:

$$\beta_{n+1}^2 \leq C\|z(0)\|_N^2(\beta_n - \beta_{n+1}).$$

The following lemma (standard in this setting, see [15, lemma 2.1] completes the proof of lemma 2.13:

Lemma 2.14. *Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and $M_0 > 0$ such that*

$$\beta_{n+1}^2 \leq M_0(\beta_n - \beta_{n+1}).$$

Then:

$$\forall n \geq 1,\ \beta_n \leq \frac{2M_0}{n}.$$

Inequality (26) implies, with lemma 2.14:

$$\|z(nT)\|^2 \leq \frac{C}{T}\|z(0)\|_N^2,$$

from which we deduce, taking into account the decay of the energy of z that (22) holds. □

Proof of lemma 2.14. Let $\alpha_n := n\beta_n$. Then:

$$\alpha_n - \alpha_{n+1} \geq \frac{\alpha_{n+1}}{n+1}\left(\frac{n}{M_0(n+1)}\alpha_{n+1} - 1\right).$$

In particular:

$$\alpha_{n+1} > 2M_0 \Rightarrow \alpha_n > \alpha_{n+1}$$

Assume there exists at least one integer n such that $\alpha_{n+1} > 2M_0$. Let N be the smallest of these integers. Then:

$$\alpha_n > \alpha_{n+1} > 2M_0,$$

which contredicts the minimality of N when $N \geq 1$ or the fact that a_0 is null when $N = 0$. □

2.4.2. *Proof of proposition (2.7).* Let V be a solution of (4), with initial data in $D(\mathcal{A}^N)$. The first step of the proof is to approach V by a solution U of the Lamé system.

It is easy to see that $(\partial_t^N v, \partial_t^{N+1} v)_{\restriction t=0}$ is in $\boldsymbol{H}_0^1 \times \boldsymbol{L}^2$. The operator Δ_e being an isomorphism from $\boldsymbol{H}_0^1 \cap \boldsymbol{H}^2$ to $\boldsymbol{L}^2$, the operator $\mathcal{L}$ is an isomorphism from $D(\mathcal{L})$ to $\boldsymbol{H}_0^1 \times \boldsymbol{L}^2$. As a consequence, we may choose (u_0, u_1) such that:

$$(u_0, u_1) \in D(\mathcal{L}^N), \quad (-\mathcal{L})^N (u_0, u_1) = (\partial_t^N v, \partial_t^{N+1} v)_{\restriction t=0}.$$

The corresponding solution of the Lamé system $U = (u, \partial_t u)$ satisfies:

$$\begin{cases} \partial_t^{N+1} u_{\restriction t=0} = \partial_t^{N+1} v_{\restriction t=0} \\ \partial_t^N u_{\restriction t=0} = \partial_t^N v_{\restriction t=0}. \end{cases}$$

Set $w = v - u$. We will first show:

$$\sum_{l=0}^{N} \|\partial_t^l w\|^2_{\boldsymbol{H}((0,T)\times\Omega)} \leq C \sum_{j=0}^{N} \|\partial_t^j \operatorname{curl} h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)}. \tag{27}$$

We have:

$$\partial_t^2 w - \Delta_e w = \kappa(\operatorname{curl} h) \wedge \boldsymbol{B}, \quad \left(\partial_t^N w, \partial_t^{N+1} w\right)_{\restriction t=0} = (0,0). \tag{28}$$

Equation (28) implies, for $l \in \{0, .., N-1\}$:

$$\|\Delta_e \partial_t^l w_{\restriction t=0}\|_{\boldsymbol{L}^2} \leq \|\partial_t^{l+2} w_{\restriction t=0}\|_{\boldsymbol{L}^2} + \|\operatorname{curl} \partial_t^l h_{\restriction t=0}\|_{\boldsymbol{L}^2}. \tag{29}$$

But any $g \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$ satisfies:

$$\|g\|_{\boldsymbol{H}_0^1} \leq \|g\|_{\boldsymbol{H}^2} \leq C \|\Delta_e g\|_{\boldsymbol{L}^2},$$

which yields, with (29):

$$\|\partial_t^l w_{\restriction t=0}\|_{\boldsymbol{H}_0^1} \leq C \left(\|\partial_t^{l+2} w_{\restriction t=0}\|_{\boldsymbol{L}^2} + \|\operatorname{curl} \partial_t^l h_{\restriction t=0}\|_{\boldsymbol{L}^2}\right). \tag{30}$$

Since $(\partial_t^N w, \partial_t^{N+1} w)_{\restriction t=0}$ is null, we deduce from (30):

$$\forall l = 0, ..., N, \quad \|\partial_t^{l+1} w_{\restriction t=0}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t^l w_{\restriction t=0}\|^2_{\boldsymbol{H}_0^1(\Omega)} \leq C \sum_{j=0}^{N-1} \|\partial_t^j \operatorname{curl} h_{\restriction t=0}\|_{\boldsymbol{L}^2(\Omega)}$$

$$\|\partial_t^{l+1} w_{\restriction t=0}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t^l w_{\restriction t=0}\|^2_{\boldsymbol{H}_0^1(\Omega)} \leq C \sum_{j=0}^{N} \|\partial_t^j \operatorname{curl} h\|_{\boldsymbol{L}^2((0,T)\times\Omega)}. \tag{31}$$

(the second ligne is a consequence of the standard trace theorem with respect to the time variable). With the energy estimates of lemma (2.4), applied to (28), we get, for any $0 \leq l \leq N$:

$$\|\partial_t^l w\|^2_{\boldsymbol{H}^1((0,T)\times\Omega)} \leq C \left(\|\partial_t^{l+1} w_{\restriction t=0}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\partial_t^l w_{\restriction t=0}\|^2_{\boldsymbol{H}_0^1(\Omega)} + \|\operatorname{curl} \partial_t^l h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)}\right),$$

which yields exactly (27).

On the other side, assumption (14) implies

$$\|u_0\|^2_{\boldsymbol{H}_0^1(\Omega)} + \|u_1\|^2_{\boldsymbol{L}^2(\Omega)} \leq C \left(\|u_0\|^2_{\boldsymbol{L}^2(\Omega)} + \|u_1\|^2_{\boldsymbol{H}^{-1}(\Omega)} + \mathcal{Q}_T^N(u)\right).$$

Hence:

$$\begin{aligned} \|v_0\|^2_{\boldsymbol{H}_0^1} + \|v_1\|^2_{\boldsymbol{L}^2} \leq & C \left(\|u_0\|^2_{\boldsymbol{H}_0^1} + \|u_1\|^2_{\boldsymbol{L}^2} + \|w_0\|^2_{\boldsymbol{H}_0^1} + \|w_1\|^2_{\boldsymbol{L}^2}\right) \\ \leq & C \left(\|u_0\|^2_{\boldsymbol{L}^2} + \|u_1\|^2_{\boldsymbol{H}^{-1}} + \mathcal{Q}_T^N(v) + \mathcal{Q}_T^N(w) + \|w_0\|^2_{\boldsymbol{H}_0^1} + \|w_1\|^2_{\boldsymbol{L}^2}\right) \\ \leq & C \left(\|v_0\|^2_{\boldsymbol{L}^2} + \|v_1\|^2_{\boldsymbol{H}^{-1}} + \mathcal{Q}_T^N(v) + \mathcal{Q}_T^N(w) + \|w_0\|^2_{\boldsymbol{H}_0^1} + \|w_1\|^2_{\boldsymbol{L}^2}\right) \end{aligned} \tag{32}$$

With (27) we get:

$$\begin{aligned}\mathcal{Q}_T^N(w) \leq& C\sum_{l=0}^{N} \|\partial_t^{l+1} w\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)}\\ \leq& C\sum_{j=0}^{N} \|\partial_t^{j} \operatorname{curl} h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)}.\end{aligned}$$

Taking into account (31) (with $l=0$) and the equation: $\operatorname{curl}(\partial_t v\wedge \boldsymbol{B}) = \partial_t h + \frac{1}{\beta}\operatorname{curl}\operatorname{curl} h$, which implies (with lemma 2.3):

$$\mathcal{Q}_T^N(v) \leq \sum_{j=0}^{N} \|\partial_t^{j} \operatorname{curl} h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)},$$

we deduce from (32):

$$\|v_0\|^2_{\boldsymbol{H}_0^1(\Omega)} + \|v_1\|^2_{\boldsymbol{L}^2(\Omega)} \leq C\left\{\|v_0\|^2_{\boldsymbol{L}^2(\Omega)} + \|v_1\|^2_{\boldsymbol{H}^{-1}(\Omega)} + \sum_{j=0}^{N} \|\partial_t^{j} \operatorname{curl} h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)}\right\}.$$

Now from lemma 2.3: $\|h_0\|^2_{\boldsymbol{L}^2} \leq C\left(\|\operatorname{curl} h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)} + \|\partial_t \operatorname{curl} h\|^2_{\boldsymbol{L}^2((0,T)\times\Omega)}\right)$.
This gives the following inequality on solutions of (4) with initial data in $D(\mathcal{A})$:

$$\|V_0\|^2_X \leq C\left\{\|V_0\|^2_{D(\mathcal{A})'} + \sum_{j=0}^{N}\left(\|\partial_t^j V(0)\|^2_X - \|\partial_t^j V(T)\|^2_X\right)\right\}.$$

It is easy to check, with the criterum given by remark 2.12, that the quadratic form:

$$Q_T(V) = \sum_{l=0}^{N}\int_\Omega |\operatorname{curl}\partial_t^l h|^2\, dy,$$

with domain $D(\mathcal{A}^N)$, is closable. All the assumptions of lemma 2.13, with $\mathcal{P} = \mathcal{A}$ hold which completes the proof of proposition 2.7.

3. Defect measures

Let N be an integer. For an open subset U of an euclidian space, we set:

$$\boldsymbol{L}^2(U) := L^2(U,\mathbb{C}^N), \quad \boldsymbol{H}^s(U) := H^s(U,\mathbb{C}^N).$$

We consider an open subset Ω of $\mathbb{R}^n$, $n\geq 1$, and a sequence (u^k) of functions on $\mathbb{R}_t\times\Omega_y$ such that:

$$u^k \underset{k\to+\infty}{\rightharpoonup} 0 \text{ in } \boldsymbol{H}^1_{\text{loc}}(\mathbb{R}\times\overline{\Omega}), \tag{33}$$

(in the sense that (φu^k) converges weakly to 0 in $\boldsymbol{H}^1(\mathbb{R}\times\overline{\Omega})$ for all $\varphi\in C_0^\infty(\mathbb{R}\times\overline{\Omega})$). We assume that every u^k is solution of a wave equation in Ω:

$$(\nu^2\partial_t^2 - \Delta)u^k = 0, \text{ in } \mathbb{R}\times\Omega. \tag{34}$$

We shall introduce in this section a measure describing, from a micro-local point of view, the defect of compactness in $\boldsymbol{H}^1$ of the sequence (u^k). This description is of fundamental importance to show the observability inequalities of the preceding section, for the Lamé system may be decomposed in two waves equation (see paragraph 3.5.1). Micro-local defect measures have been independently introduced by P. Gérard and L. Tatar [6, 16]. We shall follow the construction of N. Burq and G. Lebeau, which describes the defect of convergence up to the boundary of Ω.

We assume, for the sake of simplicity that the functions u^k are smooth, so that their traces on the boundary are always defined. In the sequel we shall always reduce to this case.

In subsection 3.1 we will give a few definitions and notations. In subsection 3.2 we will state an existence theorem of micro-local defect measures and set out their first properties. Subsection 3.3 is devoted to the propagation theorem of the measure (proved in [2]), and subsection 3.4 to some important properties of the traces of u^k on the boundary. Finally, in section 3.5, we shall apply the construction of the measure to the case of a sequence of solutions of the Lamé system.

3.1. **Notations.**

3.1.1. *Local coordinates.* Consider an open cover of Ω: $\Omega = \bigcup_{j=0}^{J} \Omega_j$, where $\overline{\Omega}_0 \subset \Omega$ and, for all $j \geq 1$, Ω_j is a small neighbourhood of a point of $\partial\Omega$, such that on Ω_j, there are geodesic normal coordinates:

$$z \in \Omega_j \mapsto (y', x_n) \in Y := Y' \times]0, l[,$$

where x_n is the distance to the boundary, and Y' an open subset of $\mathbb{R}^{n-1}$. Most objects introduced here are global objectsm but we will mainly use local coordinates. For a large part of this section we choose one of the open set Ω_j, $j \geq 1$.

Set $X := \mathbb{R} \times Y$ and denote the elements of X by:

$$x = (\underbrace{x_0, x_1, .., x_{n-1}}_{x'}, x_n), \quad x_0 = t, \quad y = (\underbrace{x_1, x_2, .., x_{n-1}}_{y'}, x_n),$$

Let:

$$\mathbb{R}^{n+1}_+ := \left\{(x', x_n) \in \mathbb{R}^{n+1},\ x_n > 0\right\}, \quad \overline{\mathbb{R}}^{n+1}_+ := \overline{\mathbb{R}^{n+1}_+}, \quad \partial X = X' \times \{0\}, \quad \overline{X} := X' \times [0, l[.$$

The set $\overline{X}$ is an open subset of $\overline{\mathbb{R}}^n_+$. Let g be the natural metric on Y, induced by the change of coordinates. In a geodesic system of coordinates, g is of the form:

$$g(y) = \begin{bmatrix} g'(y) & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathsf{g} = \det g.$$

3.1.2. *Bundles on X.* Let's consider $T^*X = X \times \mathbb{R}^{n+1}$ the cotangent bundle of X and S^*X the spherical cotangent bundle, which is defined to be the quotient

$$S^*X := (T^*X \backslash \{|\xi| = 0\}) / \mathbb{R}^*_+,$$

by the action of $\mathbb{R}^*_+ : (\lambda, \xi) \mapsto \lambda\xi$. The elements of those two bundles will be denoted by:

$$\rho = (x, \xi), \quad x \in X, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n \times \mathbb{R}, \quad \xi = (\tau, \eta).$$

There is a natural euclidian norm for the η-component of T^*X: $\|\eta\|^2 := {}^t\eta g^{-1}\eta$.

We will also consider $T^*\partial X := \partial X_{x'} \times \mathbb{R}^n_{\xi'}$ the boundary cotangent bundle and $S^*\partial X$ the associated spherical bundle.

3.1.3. *Operators in the interior of* Ω. Le S^m_i the set of matrix symboles of degree m with compact support in X, which are the functions:

$$a(x, \xi) \in C^\infty(X \times \mathbb{R}^{n+1}, M_N(\mathbb{C})),$$

whose x-projection is of compact support in X, satisfying the following estimates:

$$(35_m) \qquad \forall\alpha,\ \forall\beta,\ \exists C_{\alpha\beta},\ \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leq C_{\alpha\beta} \left(1 + |\xi|\right)^{m - |\beta|},$$

and which have a principal symbol $a_m(x, \xi)$, homogeneous function of degree m in ξ, such that $a - a_m$ satisfies (35_{m-1}) for large $|\xi|$. The operator of symbol a, $A = a(x, D)$, is defined by:

$$Av(x) := \frac{1}{(2\pi)^{n+1}} \int a(x, \xi) \hat{v}(\xi) e^{ix.\xi} d\xi.$$

In order to act on functions which are only defined in X, it is convenient to consider only the set $\mathcal{A}^m_i$ consisting of operators A which are of compact support in X, in the sense that $A = \varphi A \varphi$ for a function

$\varphi \in C_0^\infty(X)$. An operator in $\mathcal{A}_i^m$ maps a distribution in X to a compactly supported distribution in X. We shall denote by $\sigma_m(A)$ the principal symbol of an operator A of degree m.

3.1.4. *Operators near the boundary.* Let S_b^m be the set of matricial **tangential** symbols of degree m with compact support in X, defined as the functions:

$$a(x,\xi') \in C^\infty(\overline{X} \times \mathbb{R}^n, M_N(\mathbb{C})),$$

whose x-projection has compact support in $\overline{X}$, satisfying the estimations:

$$(36_m) \qquad \forall \alpha,\ \forall \beta,\ \exists C_{\alpha\beta},\ \left|\partial_x^\alpha \partial_{\xi'}^\beta a(x,\xi')\right| \leq C_{\alpha\beta}\left(1+|\xi'|\right)^{m-|\beta|},$$

and which have a principal symbol $a_m(x,\xi')$, homogeneous of degree m in ξ' and such that $a-a_m$ satisfies the inegalities (36_{m-1}) for large $|\xi'|$. We define the operator of symbol a, $A = a(x,D')$, by:

$$Av(x) := \frac{1}{(2\pi)^n}\int a(x,\xi')\hat{v}(\xi',x_n)e^{ix'\cdot\xi'}d\xi'.$$

Here, the Fourier transform of v is only taken with respect to the tangential variable x'. As in the interior case, we introduce the set $\mathcal{A}_b^m$ of tangential operators A with compact support in $\overline{X}$, i.e such that $A = \varphi A \varphi$ for a compactly supported function $\varphi \in C_0^\infty(\overline{X})$.

The set of all pseudo-differential operators of interest for us will be denoted by:

$$\mathcal{A}^m := \left\{a = A_i + A_b,\ A_i \in \mathcal{A}_i^m,\ A_b \in \mathcal{A}_n^m\right\}.$$

3.1.5. *Sobolev spaces.* Let $s \in \mathbb{R}$ and ω be an open set of $\mathbb{R}^n$. As mentionned before, we denote by $\boldsymbol{H}^s(\omega)$ the Sobolev space of vector-valued distributions (which may be defined as the set of restrictions to ω of elements of $\boldsymbol{H}^s(\mathbb{R}^n)$, endowed with the quotient norm). We also consider the space $\boldsymbol{H}^s_{\mathrm{loc}}(\omega)$, the space of vector-valued distributions such that:

$$\forall \varphi \in C_0^\infty(\omega), \quad \varphi u \in \boldsymbol{H}^s,$$

and $\boldsymbol{H}^s_{\mathrm{comp}}(\omega)$, the space of distributions in $\boldsymbol{H}^s(\omega)$ compactly supported in ω. The notation $\boldsymbol{H}^s_{\mathrm{loc}}(Z)$, will also be used when Z is not open ($Z = \mathbb{R}\times\overline{\Omega}$, or $Z = \overline{X}$, in the following natural sense:

$$u^k \underset{k\to+\infty}{\longrightarrow} u \text{ (or } = O(1)) \text{ in } \boldsymbol{H}^s_{\mathrm{loc}}(Z) \iff \forall \varphi \in C_0^\infty(Z), \quad \varphi u^k \underset{k\to+\infty}{\longrightarrow} \varphi u \ (= O(1)) \text{ in } \boldsymbol{H}^s,$$

where $C_0^\infty(Z)$ is the space of C^∞, compactly supported functions in Z. We will also consider the following spaces, suitable for boundary-value problems:

$$\boldsymbol{H}^{0,s}_{\mathrm{loc}}(\overline{X}) = L^2(0,l;\boldsymbol{H}^s_{\mathrm{loc}}(X'))$$
$$\boldsymbol{H}^{0,s}_{\mathrm{comp}}(\overline{X}) = \left\{u \in \boldsymbol{H}^{0,s}_{\mathrm{loc}}(\overline{X}),\ \exists \varphi \in C_0^\infty(\overline{X}),\ u = \varphi u\right\}.$$

Note that the elements of $\mathcal{A}_i^m$ are continuous maps:

$$\boldsymbol{H}^s_{\mathrm{loc}}(X) \longrightarrow \boldsymbol{H}^{s-m}_{\mathrm{comp}}(X),$$

and those of $\mathcal{A}_b^m$ are continuous maps:

$$\boldsymbol{H}^{0,s}_{\mathrm{loc}}\left(\overline{X}\right) \longrightarrow \boldsymbol{H}^{0,s-m}_{\mathrm{comp}}\left(\overline{X}\right).$$

It is possible to "micro-localize" convergence properties in $\boldsymbol{H}^s$ and $\boldsymbol{H}^{0,s}$:

Definition 3.1. Let $\rho \in S^*X$. The sequence (v^k) is said to be **bounded (respectively converging to** 0**) in** $\boldsymbol{H}^s_\rho$ when there exists $A \in \mathcal{A}_i^s$, whose principal symbol is invertible near ρ and such that (Av^k) is bounded in $\boldsymbol{H}^s$ (respectively converges to 0 in $\boldsymbol{H}^s$.

Let $\rho' \in S^*\partial X$. The sequence (v^k) is said to be **bounded (respectively converging to** 0**) in** $\boldsymbol{H}^{0,s}_\rho$ when there exists $A \in \mathcal{A}_i^s$, whose principal symbol is invertible near ρ' and such that (Av^k) is bounded in $\boldsymbol{H}^{0,s}$ (respectively converges to 0 in $\boldsymbol{H}^{0,s}$).

Note that, according to proposition 7.1 of the appendix, for a sequence of solutions of (34), the convergence in $\boldsymbol{H}^{0,1}$ and $\boldsymbol{H}^1$ are equivalent. The spaces $\boldsymbol{H}^{0,1}$ and the tangential operators are thus well fitted for the description of the $\boldsymbol{H}^1$ convergence of (u^k).

3.1.6. *Melrose's compressed cotangent bundle.* We shall now introduce a bundle which naturally contains as subbundles both bundles T^*X and $T^*\partial X$. For this purpose, set $T_b^*X := \overline{X} \times \mathbb{R}^{n+1}$, endowed with its canonical topology and consider:

$$\begin{array}{ccc} T^*\overline{X} & \xrightarrow{j} & T_b^*X \\ (x,\xi',\xi_n) & \longmapsto & (x,\xi',r=x_n\xi_n). \end{array}$$

The mapping j restricts to a continuous map:

$$T^*X \longrightarrow T_b^*X \cap \{x_n > 0\},$$

which identifies T^*X to a subbundle of dimension $2(n+1)$ of the interior of T_b^*X. Furthermore, the restriction of j to $x_n = 0$ defines a map from $T^*\overline{X} \cap \{x_n = 0\}$ to $T_b^*X \cap \{x_n = 0\}$, whose kernel is the set $\{\xi' = 0\}$. This clearly identifies:

$$T^*\partial X \approx (T^*\overline{X} \cap \{x_n = 0\})/\mathbb{R}_{\xi_n},$$

(quotient taken by identifiying all the points $(\tilde{x}', \tilde{\xi}', \xi_n)$, $\xi_n \in \mathbb{R}$) with a $2n$-dimensional subbundle of T_b^*X. The set of all sections of T_b^*X, with the above identifications, may be seen as the dual bundle of the bundle of all vector fields on X tangent to ∂X. It is called the compressed cotangent bundle.

We will also consider S_b^*X the spherical bundle of T_b^*X, which naturally contains the spherical bundles S^*X and $S^*\partial X$.

3.1.7. *Symbol of P and related manifolds.* The equation (34) takes the following form in local coordinates:

$$Pu^k = 0, \quad P := -\mathsf{g}^{-1/2}\partial_{x_n}\mathsf{g}^{1/2}\partial_{x_n} + Q, \tag{37}$$

where Q is a scalar tangential differential operator of degree 2. Let $q(x,\xi')$ be the principal symbol of Q, and $p(x,\xi) = \xi_n^2 + q(x,\xi')$ the principal symbol of P. They are both scalar, homogeneous polynomials of degree 2 with respect to ξ. Let:

$$\operatorname{Char} P := \left\{(x,\xi) \in T^*\overline{X},\ p(x,\xi) = 0\right\}, \quad Z := j\left(\operatorname{Char} P\right), \quad \widehat{Z} := j\left(\operatorname{Char} P\right) \cup j\left(\{x_n = 0\}\right),$$

and $S\operatorname{Char} P$, SZ, $S\widehat{Z}$ the corresponding spherical bundles.

Decompose $T^*(\partial X)$ (and $S^*(\partial X)$) into the disjoint union of the elliptic region $\mathcal{E}$, the glancing region $\mathcal{G}$ and the hyperbolic region $\mathcal{H}$:

$$\mathcal{E} := \{q_0 > 0\}, \quad \mathcal{G} := \{q_0 = 0\}, \quad \mathcal{H} := \{q_0 < 0\}.$$

3.1.8. *Global measure.* The defect measure is at first constructed in each of the preceding local coordinate systems. The objects obtained are then pieced together to $M = \mathbb{R} \times \Omega$. It is easy to define from local objects global Sobolev spaces and bundles on M, such as Melrose's compressed cotangent bundle T_b^*M. We shall use the same notations ($\operatorname{Char} P$, Z, $\widehat{Z}$, $S\operatorname{Char} P$, SZ, $S\widehat{Z}$,...) for the local and global objects. The definition of global operators is less natural in our setting. The symbol $\mathcal{A}^m$ will denote the set of operators A acting on functions on M, which are of the form:

$$A = \sum_{j=0}^{J} A_{(j)}.$$

where $A_{(0)}$ is a classical pseudo-differential operator of order m with compact support in M and each $A_{(j)}$ is an operator of the sets $\mathcal{A}^m$ defined in each system of local coordinates. The global space $\mathcal{A}^m$ depends of the coordinate patches chosen, which shall not cause any problem in the remaining of the article. For

a totally intrisic construction, we could have used Melrose's totally characteristic operators (see [7, chap 18.3]).

3.2. **Existence of the measure.**

3.2.1. *The existence theorem.* The next elementary proposition shows that for any $A \in \mathcal{A}^0$, the behaviour of Au^k in $\boldsymbol{H}^1$ only depends upon the restriction of its principal symbol to $S\widehat{Z}$:

Proposition 3.2. *Let $A_b \in \mathcal{A}_b^{-\varepsilon}$. Then:*

$$A_b u^k \underset{k\to 0}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1.$$

Let $A_i \in \mathcal{A}_i^0$, whose principal symbol vanishes on $\operatorname{Char} P$. Then:

$$A_i u^k \underset{k\to 0}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1.$$

According to proposition 3.2, it is sufficient to describe the $\boldsymbol{H}^1$ convergence of (u^k) near $S\widehat{Z}$, in the sense given by definition 3.1. Let $\mathcal{M}$ be the set of matice-valued measures on $S\widehat{Z}$, i.e. the dual space of:

$$\mathcal{C} := C_0^0\left(S\widehat{Z}, M^N(\mathbb{C})\right),$$

and $\mathcal{M}^+$ the subset of all positive measures in $\mathcal{M}$, i.e. measures μ which satisfy:

$$\forall z \in S\widehat{Z}, \quad b(z) \geq 0 \implies \langle \mu, b\rangle \geq 0.$$

($M \geq 0$ means M is positive hermitian).

Before coming to the main theorem of this paragraph, we shall introduce a technical condition on u^k:

Definition 3.3. Let the sequence (u^k) satisfies (33) and (34). We shall say that (u^k) is **regular on the boundary** when one the following equivalent assumptions is satisfied:

$$u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{H}^{1/2}_{\text{loc}}(\partial X),\ k \longrightarrow +\infty \tag{38a}$$

$$\partial_{x_n} u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{H}^{-1/2}_{\text{loc}}(\partial X),\ k \longrightarrow +\infty. \tag{38b}$$

Note that this is a very weak condition: the standard trace theorems imply conditions (38) with $O(1)$ instead of $o(1)$. All the sequences (u^k) in this work shall satisfy this condition. For the proof of the equivalence between (38a) and (38b) see [2, lemma 2.6].

Theorem 3.4. *Let u^k be such that (33), (34) and (38) hold. Then there exists a subsequence of (u^k), still denoted by (u^k), and a measure $\mu \in \mathcal{M}^+$, called* **micro-local defect measure***, such that $\mu(\mathcal{E}\cup\mathcal{H} = 0)$ and:*

$$\forall A_j \in \mathcal{A}^j,\ j \in \{1,2\},\ \lim_{k\to+\infty}\left(A_2 u^k + A_1 D_{x_n} u^k, u^k\right) = \langle \mu, \frac{a_2 + \xi_n a_1}{\tau^2}\rangle. \tag{39}$$

In (39), the notation $(.,.)$ stands for the $\boldsymbol{L}^2$-scalar product on M (in local coordinates, it is the scalar product on X using the metric $\mathsf{g}^{1/2}dy\,dt$) and μ is considered as a measure on the subset $S\operatorname{Char} P$ of $S^*\overline{X}$, using the canonical map j, which is an homeomorphism:

$$S\operatorname{Char} P \xrightarrow{j} SZ\backslash\mathcal{H}.$$

This is made possible by the fact $\mu(\mathcal{E}\cup\mathcal{H}) = 0$. In the case where (u^k) is not regular on the boundary, it is still possible to define μ, but $\mu(\mathcal{E})$ is non-null, which makes the statement of condition (39) more intricate.

Remark 3.5. The measure $\tilde{\mu} = \mu 1\!\!1_{\{x_n>0\}}$ may be seen as the standard micro-local defect measure (cf [6]) of the bounded sequence (u^k) of $\boldsymbol{H}^1_{\mathrm{loc}}(M)$. This interior measure describe the compactness defect of (u^k) in $\boldsymbol{H}^1_{\mathrm{loc}}(M)$ (in particular, it is null when (u^k) converges to 0 in this space), but not in $\boldsymbol{H}^1_{\mathrm{loc}}(\overline{M})$: $\tilde{\mu}$ vanishes when (u^k) concentrates on ∂M, even if it does not converge to 0 in $\boldsymbol{H}^1_{\mathrm{loc}}(\overline{M})$. On the other hand:

$$\varphi \in C_0^\infty(\overline{M}) \Longrightarrow \int \varphi \left|\nabla_y u^k\right|^2 dx + \int \varphi \left|\partial_t u^k\right|^2 dx \underset{k\to+\infty}{\longrightarrow} \langle \mu, \varphi \rangle .$$

Thus, μ sees all the $\boldsymbol{H}^1_{\mathrm{loc}}(\overline{M})$ density of (u^k) at infinity. More precisely, it gives a micro-local description of this density:

$$\rho \in \operatorname{supp}\mu \iff u^k \underset{k\to+\infty}{\longrightarrow} +0 \text{ in } \boldsymbol{H}^1_\rho.$$

When ρ is a boundary point, one should replace $\boldsymbol{H}^1_\rho$ by $\boldsymbol{H}^{0,1}_\rho$.

Theorem 3.4 is a new formulation, using lemma 2.7 of [2], of proposition 2.5 of this article.

3.2.2. *A sufficient condition of nullity for μ.* Let $\tilde{\rho} \in S\widehat{Z}$ an interior point and $A \in \mathcal{A}^2_i$, whose principal symbol is invertible at $\tilde{\rho}$. By elementary symbolic calculus on classical operators, it is easy to show, with formula (39):

$$Au^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^{-1}_{\tilde{\rho}} \Rightarrow \tilde{\rho} \notin \operatorname{supp}\mu. \tag{40}$$

The same statement holds in $\mathcal{G}$:

Proposition 3.6. *Consider an operator of the form:*

$$A = A_0 D^2_{x_n} + A_1 D_{x_n} + A_2,\ A_j \in \mathcal{A}^j,\ a_j := \sigma(A_j),$$

such that:

$$Au^k \underset{k\to+\infty}{\longrightarrow} 0 \textit{ in } \boldsymbol{H}^{0,-1}_{\tilde{\rho}}. \tag{41}$$

Assume that (u^k) is regular on the boundary and let $\tilde{\rho} \in \mathcal{G}$ such that $a_2(\tilde{\rho})$ is invertible. Then:

$$\tilde{\rho} \notin \operatorname{supp}\mu.$$

Remark 3.7. Proposition 3.6 is trivial when $A = A_2 \in \mathcal{A}^2$ (it is essentially the definition of $\boldsymbol{H}^{0,-1}_{\tilde{\rho}}$).

Remark 3.8. Near an hyperbolic point, it is more difficult to state proposition 3.6. Indeed it is much more relevant to study μ, in the set $\{x_n \geq \varepsilon_0 > 0\}$, near rays in and out of $\tilde{\rho}$. (see paragraph 3.4.1).

Remark 3.9. Note that according to the appendix, the convergence to 0 of (Au^k) in the space $\boldsymbol{H}^{-1}(X)$ near $\tilde{x}$ would imply (41). Furthermore, the proof of the lemma will show that assumption (41) is equivalent to:

$$(JAu^k, u^k) \underset{k\to+\infty}{\longrightarrow} 0,\ \forall J \in \mathcal{A}^0,\ \text{with support close enough to } \tilde{\rho}.$$

Proof. Let $j = \sigma(J)$. According to (40), $\mu 1\!\!1_{\{x_n>0\}}$ is null, near $\tilde{\rho}$. The same property remains to be proved on $\mu 1\!\!1_{\{x_n=0\}}$. Let:

$$J \in \mathcal{A}^0, \quad \psi \in C_0^\infty(\mathbb{R}) \text{ tel que } \psi(0) = 1, \quad J_\varepsilon := \psi\left(\frac{x_n}{\varepsilon}\right) J.$$

In view of (41) and formula (39):

$$\left\langle \mu, \psi\left(\frac{x_n}{\varepsilon}\right) j \frac{a_0\xi_n^2 + a_1\xi_n + a_2}{\tau^2} \right\rangle = \lim_{k\to+\infty} (J_\varepsilon A u^k, u^k) = 0.$$

Letting ε goes to 0, the dominated convergence theorem and the fact that ξ_n is null on the support of $\mu 1\!\!1_{\{x_n=0\}}$ give:

$$\left\langle \mu, 1\!\!1_{\mathcal{G}} \frac{j a_2}{\tau^2} \right\rangle = 0. \tag{42}$$

Let $\psi \in S^0_b$ be scalar, positive, and compactly supported near $\tilde{\rho}$ such that a_2 is invertible on the support of ψ, and choose J such that:

$$j(x,\xi') = \psi(x,\xi')a_2^{-1}\tau^2.$$

The equality (42) then shows that $\langle \mu, \mathbb{1}_{\mathcal{G}}\psi \rangle = 0$, which completes the proof using the positivity of μ. $\square$

3.3. **The propagation theorem.**

3.3.1. *The generalized bicharacteristic flow.* The characteristic curves of the hamiltonian flow of p:

$$H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$$

define a local flow on T^*X. The symbol p is homogeneous of degree 2 in ξ, so that the flow of H_p does not yield a flow on the quotient space $S^*_b X$. To get such a flow, we shall replace p by p/τ which is homogeneous of degee 1. Note that on the support of μ (where τ does not vanish), p is null, so that $\frac{1}{\tau}H_p$ and $H_{p/\tau}$ are equal. Furtermore, the integral curves of $\frac{1}{\tau}H_p$ and H_p are the same.

Let Σ be a small conic open subset of $Z = j(\operatorname{Char} P)$. Set $q_0 := q_{\restriction x_n=0}$, $q_1 := \partial_{x_n} q_{\restriction x_n=0}$ and

$$\Sigma^0 := \Sigma \cap \{x_n > 0\}$$

$$\Sigma^1 := \mathcal{H} = \Sigma \cap \{x_n = 0,\ q_0 < 0\}$$

$$\Sigma^2 := \Sigma \cap \{x_n = 0,\ q_0 = 0\ q_1 \neq 0\}$$

$$\Sigma^{k+3} := \Sigma \cap \{x_n = 0,\ q_0 = q_1 = ... = H^k_{q_0} q_1 = 0,\ H^{k+1}_{q_0}\, q_1 \neq 0\}.$$

Assume that in Σ, there is no contact of infinite order between the bicaracteristic curves of P and the boundary, which means that for a certain finite integer J:

$$\exists J \in \mathbb{N}, \quad \Sigma = \bigcup_{j \leq J} \Sigma^J. \tag{43}$$

Decompose Σ^2 in the disjoint union:

$$\Sigma^2 = \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}, \quad \mathcal{G}^{2,+} := \Sigma^2 \cap \{q_1 < 0\},\ \mathcal{G}^{2,-} := \Sigma^2 \cap \{q_1 > 0\}.$$

The set $\mathcal{G}^{2,+}$ is the set of **strictly diffractive** points and $\mathcal{G}^{2,-}$ the set of **strictly gliding** points.

Definition 3.10. Let γ be a map form a real interval I to Σ and:

$$\Gamma(s) = j^{-1}(\gamma(s)) \in S\operatorname{Char} P$$

which is defined as long as $\gamma(s) \notin \mathcal{H}$. Such a map $\gamma(s) = (x(s), \xi(s))$ is called a **ray**, or a **general bicharacteristic curve** when γ is continuous from I to Σ and for all s_0 in I:

- if $x_n(s_0) > 0$, Γ is differentiable in s_0 and:

$$\Gamma'(s_0) = \frac{1}{\tau} H_p \Gamma(s_0);$$

- if $\gamma(s_0) \in \mathcal{H} \cup \mathcal{G}^{2,+}$,

$$\exists \varepsilon > 0, \quad \forall s \in]s_0 - \varepsilon, s_0[\cup]s_0, s_0 + \varepsilon[, \quad x_n(s) > 0;$$

- if $\gamma(s_0) \in \mathcal{G} \backslash \mathcal{G}^{2,+}$, Γ is well defined and differentiable near s_0 and:

$$\Gamma'(s_0) = \frac{1}{\tau} H_{q_0} \Gamma(s_0).$$

  (Thus, if γ stays in this region, its spatial projection is a geodesic of the boundary.)

Under the assumption (43), R. Melrose and J. Sjöstrand have shown that for any $\rho \in \Sigma$, there exists an unique maximal ray γ taking values in Σ such that $\gamma(0) = \rho$ (cf [10], [7, chap 24.3]). In the sequel, we shall denote by $\phi(s,\rho)$ the resulting flow (satisfying $\phi(0,\rho) = \rho$). The function p/τ being homogeneous of degree 1 in ξ, the flow ϕ passes to the quotient and defines a flow on $\Sigma/\mathbb{R}^*_+$.

3.3.2. *The uniform Lopatinsky conditions.*

Notations. Let S^m_∂ be the set of symbols $a(x',\xi')$ of pseudo-differential operators on ∂X, with compact support in x', with principal symbol homogenous of degree m in ξ', and $\mathcal{A}^m_\partial$ the set of corresponding compactly supported pseudo-differential operators (cf paragraph 3.1.3 for precise definitions). The Sobolev spaces on ∂X, defined as those on X, shall be denoted by $\boldsymbol{H}^s_\partial$, $\boldsymbol{H}^s_{\mathrm{loc},\partial}$, $\boldsymbol{H}^s_{\tilde\rho,\partial}$.

An approximate pseudo-differential equation on the traces of u^k is said to satisfy Lopatinsky conditions when it is independent of the equation $Pu^k=0$. Precisely:

Definition 3.11. Under the assumptions (33) and (34), the sequence (u^k) is said to satisfy **uniform Lopatinsky** boundary conditions near $\tilde\rho\in S^*\partial X$ when:

- if $\tilde\rho\in\mathcal{G}$, $\exists B_{-1}\in\mathcal{A}^{-1}_\partial$ such that:

$$\begin{cases} u^k_{\restriction x_n=0} = B_{-1}\left(D_{x_n}u^k_{\restriction x_n=0}\right)+h^k \\ h^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}; \end{cases} \tag{44}$$

- if $\tilde\rho\in\mathcal{H}$, $\exists B_0\in\mathcal{A}^0_\partial$ such that:

$$\begin{cases} D_{x_n}u^k_{\restriction x_n=0}-\Lambda u^k_{\restriction x_n=0} = B_0\left(D_{x_n}u^k_{\restriction x_n=0}+\Lambda u^k_{\restriction x_n=0}\right)+h^k \\ \sigma(B_0) \text{ invertible near } \tilde\rho \\ h^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial} \end{cases} \tag{44'}$$

$$\Lambda\in\mathcal{A}^1_\partial,\quad \sigma(\Lambda)=\sqrt{q_0(x',\xi')}=\sqrt{\nu^2\tau^2-\|\eta'\|^2}\text{ near }\tilde\rho.$$

($\|\eta'\|^2={}^t\eta' g'^{-1}\eta'$ is the natural euclidian norm in the local coordinate system).

Examples.

- The Dirichlet boundary condition, $u^k_{\restriction x_n=0}=0$, or more generally a pseudo-differential boundary condition of the form:

$$\begin{gathered} u^k_{\restriction x_n=0}=B_{-1}D_{x_n}u^k_{\restriction x_n=0}+h^k,\quad B_{-1}\in\mathcal{A}^{-1} \\ h^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}, \end{gathered} \tag{45}$$

  where the eigenvalues of $\sigma(B_{-1})$ are all pure imaginary numbers near $\tilde\rho$, is an uniform Lopatinsky boundary condition, whether $\tilde\rho$ is glancing or hyperbolic. In the glancing case (45) corresponds exactly to the definition (44) and in the hyperbolic case, both operators $\mathrm{Id}-\Lambda B_{-1}$ and $\mathrm{Id}+\Lambda B_{-1}$ are elliptic in $\tilde\rho$, and it is easy to show (44'), taking $B_0=(\mathrm{Id}-\Lambda B_{-1})(\mathrm{Id}+\Lambda B_{-1})^{-1}$ (where $(...)^{-1}$ stands for a parametrix near $\tilde\rho$).
- Neumann condition:

$$D_{x_n}u^k_{\restriction x_n=0}\underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}$$

  is an uniform Lopatinsky condition near any $\tilde\rho\notin\mathcal{G}$.
- Boundary conditions which are not Lopatinsky conditions in the hyperbolic region are described in paragraph 3.4.1.

In the glancing case, a boundary condition of the form (44) locally implies better estimates than the standard ones on the traces of u^k, which we shall state in proposition 3.19. This shows in particular that (u^k) is regular near the boundary, and that $\mu(\mathcal{E})=0$. As a consequence, the bicharacteristic flow is defined μ-almost everywhere.

The set $\mathcal{G}^{2,+}$ is transverse to the bicharacteristic flow. The next result is necessary to the propagation of μ by the flow, which is treated in the next paragraph.

Lemma 3.12. *Suppose that on Σ, u^k satisfies uniform Lopatinsky boundary conditions. Then:*

$$\mu\left(\mathcal{G}^{2,+}\cap\Sigma\right)=0.$$

3.3.3. *The propagation theorem.* When an uniform Lopatinsky condition holds, μ propagates along the integral curves of the bicharacteristic flow. In the hyperbolic region, there is a jump (which depends upon the boundary condition). We shall only state a propagation theorem for the support of μ, without giving a complete description of the propagation of μ.

Theorem 3.13. *Let $\tilde{\rho} \in \mathcal{H} \cap \mathcal{G}$ such that (u^k) satisfies Lopatinsky boundary conditions. Consider a small conic open neighbourhood Σ of $\tilde{\rho}$ in $S\hat{Z}$ such that on Σ, (44) (or (44')) holds. Then the support of μ is, in Σ invariant by the bicharacteristic flow.*

(cf [2, chap. 3.3, th.1])

In other terms, if $\rho \in \Sigma$ is on the support of μ, so is the entire bicharacteristic passing through ρ in Σ.

Remark 3.14. Inside M, theorem 3.13 is an easy consequence of the transport equation on μ:

$$\big\langle \mu, \{p/\tau, a\} \big\rangle, \quad a \in C_0^\infty \left(Z \cap \{x_n > 0\}\right), \tag{46}$$

which may be immediately derived, using symbolic calculus, from the elementary property:

$$\lim_{k \to +\infty} \left(A_1 P u^k - P A_1 u^k, u^k\right) = 0, \quad A_1 \in \mathcal{A}_i^1, \tag{47}$$

obtained by integration by parts with the equation (37). Near a boundary point, property (47), with $A_1 \in \mathcal{A}_b^1$, still holds with an additional boundary term. Consequently, (46) holds only for a certain class of function $a \in C_0^\infty(SZ)$, satisfying a particular boundary condition on $\{x_n = 0\}$ (condition chosen to kill the boundary terms when k tends to ∞). The proof of the propagation theorem, which is fairly technical, uses (46), and near strictly diffractive points, lemma 3.12. The boundary condition on a gives the exact value of the jump in the hyperbolic region. See [2, par. 3] for details.

3.4. **Estimates on traces.** We now state precise properties of the traces of u^k in the hyperbolic, elliptic and glancing regions, which are one of the main tools of the proofs of the following sections. Those results are fairly classical, and we only shall give a proof (in the appendix) for the glancing case. See [2, appendix] for proofs in the hyperbolic and elliptic cases. In this paragraph, we shall always assume (u^k) satisfies (33) and (34).

3.4.1. *Hyperbolic region.* Near an hyperbolic point, one gains without any boundary condition, half a derivative in comparision with the standard traces theorem.

Proposition 3.15. *Let $\tilde{\rho} \in \mathcal{H}$. Then:*

$$u^k_{\restriction x_n=0} \underset{k\to\infty}{=} O(1) \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}$$
$$\partial_{x_n} u^k_{\restriction x_n=0} \underset{k\to\infty}{=} O(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

In view of the propagation theorem in the interior of M, the support of μ is, near $\tilde{\rho}$, the union of incoming rays (integral curves of $H_{p/\tau}$ along which $\xi_n < 0$) and outgoing rays (integral curves of $H_{p/\tau}$ along which $\xi_n > 0$). When the sequence satisfies uniform Lopatinsky conditions, theorem 3.13 is equivalent to the fact that if an incoming (respectively outgoing) ray is in the support of μ, so is the outgoing (respectively incoming) ray passing through the same hyperbolic point. In the opposite case where the support of μ contains, locally, only incoming (or only outgoing) rays, one gets a boundary condition which is in a certain sense "orthogonal" to Lopatinsky uniform conditions:

Proposition 3.16. *Assume that near $\tilde{\rho} \in \mathcal{H}$, on the support of μ, $\xi_n > 0$. Then:*

$$D_{x_n} u^k_{\restriction x_n=0} + \Lambda u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}$$
$$\Lambda \in \mathcal{A}^1_\partial, \quad \sigma_1(\Lambda) = \sqrt{\nu^2\tau^2 - \|\eta'\|^2}$$

On the other hand, if near $\tilde{\rho} \in \mathcal{H}$, on the support of μ, $\xi_n < 0$, then:

$$D_{x_n} u^k_{\restriction x_n=0} - \Lambda u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

In particular, if μ is null near $\tilde\rho$,

$$D_{x_n} u^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}, \quad u^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}.$$

3.4.2. *In the elliptic region.* In $\mathcal{E}$, the equation (37) implies a pseudo-differential traces equation on u^k:

Proposition 3.17. *Let $\tilde\rho \in \mathcal{E}$ and (u^k) satisfy the assumptions of theorem 3.4. Let $M > 0$. Then:*

$$D_{x_n} u^k_{\restriction x_n=0} + \Xi u^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H^M_{\tilde\rho} \tag{48}$$

$$\Xi \in \mathcal{A}^1, \quad \sigma_1(\Xi) = i\sqrt{q_0} = i\sqrt{\|\eta'\|^2 - \nu^2\tau^2} \text{ near } \tilde\rho.$$

In particular, if a boundary condition independent of (48) holds on u^k near ρ_0 (such a condtion is called as in the glancing and hyperbolic cases an uniform Lopatinsky condition), the traces of u^k converge to 0 in appropriate Sobolev spaces $H^M_{\tilde\rho}$. Proposition 3.17 still holds in a much more general case, for example if P is replaced by a non-scalar operator $\mathbf{P}$. The principal symbol of Ξ depends again upon the principal symbol of $\mathbf{P}$. In the next proposition, we only state a consequence of this fact when (u^k) satisfies Dirichlet boundary conditions (which are of uniform Lopatinsky type).

Proposition 3.18. *Let:*

$$\mathbf{P} := D^2_{x_n} + \mathbf{Q}_1 D_{x_n} + \mathbf{Q}_2,$$

where each $\mathbf{Q_j}$ is a matricial pseudo-differential operator of degree j, with principal symbols $\mathbf{q}_j$. Let:

$$\mathbf{p}(x,\xi) := \mathbf{q}_2 + \xi_n \mathbf{q}_1$$

be the principal symbol of $\mathbf{P}$, and $\tilde\rho = (\tilde x', \tilde\xi')$ be a point of $S^*\partial X$ such that the matrix $\mathbf{p}(\tilde x', 0, \tilde\xi', \xi_n)$ is invertible for any real number ξ_n. Consider a sequence (u^k), weakly converging to 0 in $\boldsymbol{H}^1_{\text{loc}}(\overline{X})$ and satisfying:

$$\mathbf{P}u^k = 0, \quad u^k_{\restriction x_n=0} = 0.$$

Then for all M:

$$\partial_{x_n} u^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^M_{\tilde\rho,\partial}.$$

We shall later apply the preceding proposition in the elliptic zone ($\mathcal{E}_T \cap \mathcal{E}_L$ with notations of paragraph 3.5.2) of the Lamé operator $\partial_t^2 - \Delta_e$.

3.4.3. *In the glancing region.* The strong results of the two preceding paragraphs do not hold in the neighbourhood of a glancing point. In this case, one need boundary conditions to get further estimation than the standard traces theorem with loss of one half-derivative. In the case of Lopatinsky boundary conditions, the results are similar to those of the hyperbolic region.

Proposition 3.19. a) *Let $\tilde\rho \in \mathcal{G}$. Assume that (u^k) satisfies Lopatinsky uniform boundary conditions near $\tilde\rho$. Then:*

$$u^k_{\restriction x_n=0} = O(1) \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}, \quad \partial_{x_n} u^k_{\restriction x_n=0} = O(1) \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}. \tag{49}$$

Furthermore, if μ vanishes near $\tilde\rho$, then:

$$u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}, \quad \partial_{x_n} u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}. \tag{50}$$

b) *Assume:*

$$u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}, \quad \partial_{x_n} u^k_{\restriction x_n=0} = o(1) \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}, \tag{51}$$

and that every $\tilde\rho \in \mathcal{G}$ is not diffractive, in the sense that at least one off the two half-bicharacteristic passing through $\tilde\rho$ stays in $\partial\Omega$ near $\tilde\rho$. Then $\mu = 0$ near $\tilde\rho$.

Remark 3.20. As seen in propositions 3.15 and 3.16, point **a)** holds in the hyperbolic case, where no boundary condition is required.

The proof of proposition 3.19 is given in the appendix.

3.5. **The Lamé system.** This subsection is devoted to the Lamé system with Dirichlet boundary conditions on an open bounded subset Ω of $\mathbb{R}^3$:

$$\begin{cases} \partial_t^2 u - \Delta_e u = 0, \ (t,y) \in \mathbb{R}\times\Omega \\ u_{\restriction \partial\Omega} = 0 \\ u_{\restriction t=0} = u_0 \in \boldsymbol{H}_0^1, \ \partial_t u_{\restriction t=0} = u_1 \in \boldsymbol{L}^2. \end{cases} \tag{52}$$

In paragraph 3.5.1, (52) is decomposed into two wave equations. In paragraph 3.5.2, we shall introduce the defect measures associated to these equations. Next paragraphs are devoted to a few elementary properties of these measures.

3.5.1. *Transversal and longitudinal waves.* The natural energy:

$$E(t) := \frac{1}{2}\int_\Omega \left(|\partial_t u|^2 + \mu|\nabla u|^2 + (\lambda+\mu)|\operatorname{div} u|^2\right) \mathrm{d}y$$

is time-invariant. Let E_0 be its constant value. The nest classical proposition is proved, for example, in [2].

Proposition 3.21 (Decomposition of the Lamé system)**.** *There exists a constant $C>0$ such that for every solution u of (52), there exists:*

$$u_T \in \boldsymbol{H}^1_{\mathrm{loc}}\left(\overline{M}\right), \ u_L \in \boldsymbol{H}^1_{\mathrm{loc}}\left(\overline{M}\right),$$

such that:

(1) $u = u_T + u_L, \ \operatorname{div} u_T = 0, \ \operatorname{curl} u_L = 0.$
(2) $(\partial_t^2 - c_T^2\Delta)u_T = 0,$ *where* $c_T^2 := \mu.$
(3) $(\partial_t^2 - c_L^2\Delta)u_L = 0,$ *where* $c_L^2 := \lambda + 2\mu.$
(4) *For every bounded interval I of $\mathbb{R}$, of length $|I|$:*

$$\|u_L\|^2_{\boldsymbol{H}^1(I\times\Omega)} + \|u_T\|^2_{\boldsymbol{H}^1(I\times\Omega)} \leq C|I|E_0.$$

(5) *If u is in the space vector generated by a finite number of eigenfunction of $\mathcal{L}$, then:*

$$u_T \in C^\infty(\overline{M}), \quad u_L \in C^\infty(\overline{M}).$$

Definition 3.22. The function u_T is called **transversal wave**, and the function u_L **longitudinal**.

Remark 3.23. In the sequel we shall often reduce the longitudinal wave to a scalar function, writing $u_L = \nabla\varphi$, with:

$$\varphi \in \boldsymbol{H}^2_{\mathrm{loc}}\left(\overline{M}\right), \quad \|\varphi\|_{\boldsymbol{H}^2(I\times\Omega)} \leq C|I|E_0, \quad (\partial_t^2 - c_L^2\Delta)\varphi = 0.$$

3.5.2. *Measures.* Let (u^k) be a sequence of solutions of the Lamé system with:

$$(u_0^k, u_1^k) \underset{k\to+\infty}{\rightharpoonup} 0 \text{ in } \boldsymbol{H}_0^1(\Omega)\times\boldsymbol{L}^2(\Omega).$$

In view of the continuity of the map introduce in proposition 3.21:

$$\begin{array}{ccc} (u_0,u_1) & \longmapsto & (u_L,u_T) \\ \boldsymbol{H}_0^1\times\boldsymbol{L}^2 & \longrightarrow & \left(\boldsymbol{H}^1(I\times\Omega)\right)^2, \end{array}$$

where I is a bounded interval, both sequences (u_T^k) and (u_L^k) weakly converge to 0 in $\boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}\times\overline{\Omega})$. Likewise, the sequence $(\partial_t\varphi^k)$ weakly converges to 0 in $H^1_{\mathrm{loc}}(\mathbb{R}\times\overline{\Omega})$ (it is more convenient to consider a derivative of φ in order to work in a H^1 space, for which the defect measures introduced here are well fitted).

Lemma 3.24. *The sequences (u_T^k), (u_L^k), $(\partial_t\varphi^k)$ are regular on the boundary.*

(cf [2, lemme 4.2])

Notations. Let:

- μ_T, μ_L and μ be the defect measures respectively associated (up to a subsequence) to (u_T^k), (u_L^k), and $(\partial_t \varphi^k)$ by theorem 3.4;
- $\mathcal{H}_T$, $\mathcal{H}_L$, $\mathcal{G}_T$, $\mathcal{G}_L$, $\mathcal{E}_T$, $\mathcal{E}_L$ the hyperbolic, glancing and elliptic region of the transversal and longitudinal waves.

All the calculation shall be carried out in one of the $J+1$ local coordinate systems choosen in the beginning of this section. We shall make a distinction between the spaces of scalar operators: $\mathcal{A}^m$, $\mathcal{A}^m_\partial$ (defined in paragraphs 3.1.3, 3.1.4, 3.1.8, with $N=1$), and the spaces of matricial operators $\boldsymbol{\mathcal{A}}^m$, $\boldsymbol{\mathcal{A}}^m_\partial$ (with $N=3$).

The notation $x=(y,t)$ always refers to local coordinates. When a distinction is necessary, we shall write coordinates on Ω before the change of variables $z=(z_1,z_2,z_3)$. This global system of coordinates has been chosen so that the magnetic field is vertical: $\boldsymbol{B}=(B,0,0)$.

Remark 3.25. One may identify μ_T and μ_L to measures on $S_b^*\left(\mathbb{R}\times\overline{\Omega}\right)$ with values endomorphism of $\mathbb{C}^3$.

Remark 3.26. Condition $c_L^2 \neq c_T^2$ means that the intersection of $\mathcal{G}_T$ and $\mathcal{G}_L$ is empty.

3.5.3. *Link between μ and μ_L.* Let χ be the local diffeomorphism from global spatial coordinates (z_1,z_2,z_3) to local coordinates:

$$z=\chi(y), \quad {}^t\chi'(y)\zeta=\eta.$$

Proposition 3.27.

$$\forall a \in C_0^0\left(S_b^*(\mathbb{R}\times\overline{\Omega}), M^3(\mathbb{C})\right),\ \left\langle \mu, \frac{{}^t\zeta a \zeta}{\tau^2}\right\rangle = \langle \mu_L, a\rangle.$$

In particular measures μ and μ_L have the same supports.

Proof. Let $A_j \in \boldsymbol{\mathcal{A}}^j$, $j=1,2$, $\quad A:=A_{-1}D_{x_n}+A_0$.
Set: $I_k := (A\partial_t u_L^k, \partial_t u_L^k)$.
On one hand:

$$I_k = -(\partial_t A \partial_t u_L^k, u_L^k) \underset{k\to+\infty}{\longrightarrow} \langle \mu_L, a\rangle\,.$$

On the other hand:

$$I_k = -(\operatorname{div} A \nabla \partial_t \varphi^k, \partial_t\varphi^k) + o(1) \underset{k\to+\infty}{\longrightarrow} \left\langle \mu_L, \frac{{}^t\zeta a\zeta}{\tau^2}\right\rangle.$$

The boundary terms of this preceding integration by parts converge to 0 according to lemma 3.24. This implies proposition 3.27 when a is of the form $\xi_n a^{-1} + a_0$, and then by a density argument for any a. ☐

3.5.4. *Polarization of μ_T and μ_L.* Let π bet the orthogonal projection in $\mathbb{C}^3$ on the line generated by ζ, and $\pi_\perp$ the orthogonal projection on the plane normal to ζ.

$$\pi(V) := |\zeta|^{-2}({}^t\zeta.V)\zeta, \quad \pi_\perp := \operatorname{Id}_{\mathbb{C}^3} - \pi. \tag{53}$$

Projectors π and $\pi_\perp$ are defined by formulas (53), on $S^*\overline{X}$.

Proposition 3.28. *The measure μ_L is polarized along the direction of propagation, and μ_T orthogonally to this direction:*

$$\mu_L = \pi\mu_L\pi, \quad \mu_T = \pi_\perp \mu_T \pi_\perp.$$

Proof. The statement on μ_L is an immediate consequence of proposition 3.27. To show the statement on μ_T, take $A_0 \in \mathcal{A}_0$. The nullity of $\operatorname{div} u_T^k$ implies:

$$0 = (A_0 \nabla \operatorname{div} u_T^k, u_T^k) \underset{k\to+\infty}{\longrightarrow} \left\langle \mu_T, \frac{a_0 \zeta\, {}^t\zeta}{\tau^2}\right\rangle.$$

Thus:

$$<\mu_T, a\pi>=0, \quad <\mu_T, a\pi_\perp> = <\mu_T, a>\,.$$

The symmetry of μ_T completes the proof. □

Remark 3.29. To get more intrisic formulations of the preceding results, i.e. statements where the two coordinate system do not mix, one should have considered u_L^k and u_T^k as section of the tangent space $T\Omega$, and defined measures with values endomorphism of $T\Omega$ (instead of endomorphism of $\mathbb{C}^3$).

3.5.5. *A decoupling lemma.* The next result, converting an approximate differential equation on u^k into two equations on u_L^k and u_T^k, is of crucial importance in the sequel. As before, $(.,.)$ stand for the $\boldsymbol{L}^2$ scalar product on $\mathbb{R}\times\Omega$.

Lemma 3.30 (decoupling lemma)**.** *Let A be a pseudo-differential operator of order 2 of the following form:*

$$A=\sum_{j=-M}^{2} A_j\partial_{x_n}^{2-j}+A_i \tag{54}$$

$$A_i\in\boldsymbol{\mathcal{A}}_i^2,\quad A_j\in\boldsymbol{\mathcal{A}}^j.$$

Then:

$$\lim_{k\to+\infty}(Au_T^k,u_L^k)=\lim_{k\to+\infty}(Au_L^k,u_T^k)=0. \tag{55}$$

Let A be a $(1,3)$ matrix of pseudo-differential operators, with coefficients of the form (54), but with **scalar** *operators. Then:*

$$\lim_{k\to+\infty}(Au_T^k,\partial_t\varphi^k)=0. \tag{56}$$

Proof. We shall only prove the convergence to 0 of (Au_T^k,u_L^k). The proof of rest of the lemma is very much the same. We may obviously assume that the oprators A_j have compact support in one of the local coordinate system introduced in paragraph 3.1.8. In view of the equations:

$$-\mathsf{g}^{-1/2}\partial_{x_n}\mathsf{g}^{1/2}\partial_{x_n}u_T^k+Q_Tu_T^k=0,\quad -\mathsf{g}^{-1/2}\partial_{x_n}\mathsf{g}^{1/2}\partial_{x_n}u_L^k+Q_Lu_L^k=0,$$

where Q_T and Q_L are tangential, it is sufficient to prove the lemma in the cases $j=1,2$ and in the interior case.

First case: $A\in\mathcal{A}^i$.

In view of: $\nu_L\neq\nu_T$ it is easy to construct two operators:

$$\Psi_T,\Psi_L\in\mathcal{A}_i^0,\quad (\Psi_T+\Psi_L)_{\restriction U}=\mathrm{Id}$$

$$\operatorname{supp}\sigma_0(\Psi_T)\cap\mathcal{Z}_L=\operatorname{supp}\sigma_0(\Psi_L)\cap\mathcal{Z}_T=\emptyset,$$

where U is a small open subset of Ω such that there exists a function $\varphi\in C^\infty(U)$ satisfying $\varphi A\varphi=A$. Writing:

$$A=A\varphi=A\Psi_L\varphi+A\Psi_T\varphi,$$

we may assume that the principal symbol of A does not intersect $\mathcal{Z}_T$ (or does not intersect $\mathcal{Z}_L$). For such operators, (55) holds trivially. For example, in the first case we have:

$$Au_T^k=O(1)\text{ in }\boldsymbol{L}^2.$$

Second case: $A\in\mathcal{A}_\partial^2$.

We know that the support of $\mu_T\mathbb{1}_{\{x_n=0\}}$ is included in $\mathcal{G}_T$ and the support of $\mu_L\mathbb{1}_{\{x_n=0\}}$ in $\mathcal{G}_L$. As a consequence, we may write $A=A(\Theta_T+\Theta_L)$ where Θ_T and Θ_L are tangential operators of degree 0 such that:

$$\operatorname{supp}\sigma_0(\Theta_T)\cap\mathcal{G}_L=\operatorname{supp}\sigma(\Theta_L)\cap\mathcal{G}_T=\emptyset.$$

We may thus assume that the support of the principal symbol a_2 of A is disjoint from one of the two glancing sets, say $\mathcal{G}_T$. We have:

$$\big(Au_T^k,u_L^k\big)=\big(\chi(x_n/\varepsilon)Au_T^k,u_L^k\big)+\big((1-\chi(x_n/\varepsilon))Au_T^k,u_L^k\big).$$

where χ is a compactly supported function in $\mathbb{R}$ equal to 1 near the origin. We first fix ε and let k tend to ∞. The second term of the sum tends to 0 in view of the preceding case. As for the first term, we have, u_L^k being bounded in $\boldsymbol{H}^1$:

$$\begin{aligned}\left|\left(\chi(x_n/\varepsilon)Au_T^k,u_L^k\right)\right| \leq& C\|\chi(x_n/\varepsilon)Au_T^k\|_{L^2(0,l,\boldsymbol{H}^{-1}(X'))}\|u_L^k\|_{L^2(0,l,\boldsymbol{H}^1(X'))}\\ \leq& C\|\Lambda'_{-1}\chi(x_n/\varepsilon)Au_T^k\|_{\boldsymbol{L}^2}+o(1),\ k\to+\infty,\end{aligned}$$

where $\Lambda'_{-1}\in\mathcal{A}_\partial^{-1}$, with principal symbol equal to $\|\xi'\|^{-1}$ near the support of A. We have:

$$\|\Lambda'_{-1}\chi(x_n/\varepsilon)A\,u_T^k\|^2_{\boldsymbol{L}^2}\underset{k\to+\infty}{\longrightarrow}\left\langle\mu_T,\frac{(\chi(x_n/\varepsilon))^2\,|a_2|^2}{\tau^2\|\xi'\|^2}\right\rangle$$

$$\limsup_{k\to+\infty}\left|\left(Au_T^k,u_L^k\right)\right|\leq\left\langle\mu_T,\frac{(\chi(x_n/\varepsilon))^2\,|a_2|^2}{\tau^2\|\xi'\|^2}\right\rangle.$$

When ε goes to 0, the right side of this inequality converges (by the dominated convergence theorem) to:

$$\left\langle\mu_T,\mathbb{1}_{\{x_n=0\}}|a_2|^2\tau^{-2}\|\xi'\|^{-2}\right\rangle,$$

which is null, because $\mathcal{G}_T$ and the support of a_2 are disjoint.

Third case: $A=A_1D_{x_n}$, A_1 tangential.

As in the preceding case, we may assume that $\sigma_1(A_1)$ is disjoint with one of the two glancing sets, say $\mathcal{G}_T$. Then:

$$\left(Au_T^k,u_L^k\right)=\left(\chi(x_n/\varepsilon)Au_T^k,u_L^k\right)+\left((1-\chi(x_n/\varepsilon))Au_T^k,u_L^k\right)$$

The second term converges to zero when $k{\to}\infty$ for the same reasons as in the case $A\in\mathcal{A}_i^2$. The first term may be written:

$$\begin{aligned}\left(\chi(x_n/\varepsilon)A_1D_{x_n}u_T^k,u_L^k\right)&=\left(D_{x_n}(\chi(x_n/\varepsilon)A_1u_T^k),u_L^k\right)+\left(R_\varepsilon u_T^k,u_L^k\right),\ R_\varepsilon\in\boldsymbol{\mathcal{A}}^0\\&=\left((\chi(x_n/\varepsilon)A_1u_T^k),D_{x_n}u_L^k\right)+\left(R_\varepsilon u_T^k,u_L^k\right)+\{\textit{boundary terms}\}\end{aligned}$$

The boundary terms tend to zero when k tends to infinity because (u_T^k) and (u_L^k) are regular on the boundary. The proof may be completed as in the preceding case, letting k go to infinity then ε go to zero. □

4. Sufficient condition

If γ is a $\boldsymbol{B}$-resistant ray defined on a real interval $]a,b[$, we shall call life-length the positive quantity $|t(b)-t(a)|$. In this section we use tools of the preceding section to prove the following:

Proposition 4.1. *Let $T>0$. Assume that every $\boldsymbol{B}$-resistant ray in Ω is of life-length strictly less than T. Then there exists $C>0$ such that for evey solution of the Lamé system (52):*

$$\|u_0\|^2_{\boldsymbol{H}_0^1}+\|u_1\|^2_{\boldsymbol{L}^2}\leq C\left(\|\mathrm{curl}\,(\partial_tu\wedge\boldsymbol{B})\|^2_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}+\|u_0\|^2_{\boldsymbol{L}^2}+\|u_1\|^2_{\boldsymbol{H}^{-1}}\right)\tag{57}$$

Inequality (57) is the sufficient condition (11) for uniform decay stated in point a) of proposition 2.5. Proposition 4.1 thus completes the proof of the sufficient condition of theorem 1, namely that the non-existence of arbitrarily large $\boldsymbol{B}$-resistant rays on Ω implies the uniform decay of the energy for solutions of the system of magnetoelasticity.

To show (57), we shall argue by contradiction, considering the defect measures $\mu_{T,L}$ of subsection 3.5.2 associated to a sequence (u^k) which contredicts (57) (cf subsection 4.1). The bound on $\mathrm{curl}\,(\partial_tu^k\wedge\boldsymbol{B})$ given by the negation of (57) implies a strong condition on the supports of these measures (see subsection 4.2). In subsection 4.3, we make use of this condition, together with propagation arguments near the boundary of Ω. Subsection 4.4 completes the proof, using the assumption of non-existence of $\boldsymbol{B}$-resistant rays of life-length larger than T.

4.1. **Introduction of measures.** Assume that (57) does not hold. Then there exists a sequence (u^k) of solutions of the Lamé system such that:

$$1 = \|u_0^k\|^2_{\boldsymbol{H}_0^1} + \|u_1^k\|^2_{\boldsymbol{L}^2} > k\Big(\|\mathrm{curl}\,(\partial_t u^k \wedge \boldsymbol{B})\|^2_{\boldsymbol{H}^{-1}((0,T)\times\Omega)} + \|u_0^k\|^2_{\boldsymbol{L}^2} + \|u_1^k\|^2_{\boldsymbol{H}^{-1}}\Big). \tag{58}$$

Up to the extraction of a subsequence, one may assume that (u_0^k, u_1^k) weakly converges in $\boldsymbol{H}_0^1 \times \boldsymbol{L}^2$. Inequality (58) implies that its weak limit is 0. We may thus introduce the defect measures μ_T, μ_L and μ of paragraph 3.5.2, associated to the sequences (u_T^k), (u_L^k) and $(\partial_t\varphi^k)$. To contredict (58), we need to show that these measures are null. Note that (58) implies:

$$\mathrm{curl}\,(\partial_t u^k \wedge \boldsymbol{B}) \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H^{-1}((0,T)\times\Omega). \tag{59}$$

Remark 4.2. By a density argument, it suffices to show (57) with (u_0, u_1) generated by a finite number of eigenfunctions of $\mathcal{L}$. We may thus assume, that u_L^k and u_T^k are C^∞.

Remark 4.3. We will indeed show a more precise statement than proposition 4.1 namely that if (u^k) is a sequence of solutions of the Lamé system converging weakly to 0 in the energy space and satisfying (59) then the set $(\mathrm{supp}\,\mu_T \cup \mathrm{supp}\,\mu_L) \cap \{t \in (0,T)\}$ a an union of $\boldsymbol{B}$-resistant rays of length T.

4.2. **Condition on the supports.** We may see $\boldsymbol{B}$ as a vector field on Ω, i.e. a section of $T\Omega$. To avoid confusions, the magnetic field considered as a vector field shall be refered as $\overrightarrow{B}$. In a local coordinate system, if:

$$\chi : U \subset \overline{\Omega} \longrightarrow \overline{\mathbb{R}}^n_+$$

is the change of coordinates, and χ' its differential, $\overrightarrow{B}$ is equal to $\chi'\boldsymbol{B}$. Notation $\boldsymbol{B}$ shall always refer to the vector of $\mathbb{R}^3$ of coordinates $(B,0,0)$. As before, (z_1,z_2,z_3) refers to the global spatial coordinates on Ω, before the change of variable.

Lemma 4.4. *Assume (59). Then, on the interval $(0,T)$, the transversal measure charges set of all points whose direction of propagation is orthogonal to $\boldsymbol{B}$ and the longitudinal measure charges set of all points whose direction of propagation is parallel to $\boldsymbol{B}$.*

$$\mu_T \mathbb{1}_{(0,T)} = \mu_T \mathbb{1}_{(0,T)} \mathbb{1}_{\overrightarrow{B}^\perp}, \quad \overrightarrow{B}^\perp := \{(t,y,\tau,\eta);\ {}^t\overrightarrow{B}\eta = 0\} \tag{60}$$

$$\mu_L \mathbb{1}_{(0,T)} = \mu_L \mathbb{1}_{(0,T)} \mathbb{1}_{\overrightarrow{B}^{//}}, \quad \overrightarrow{B}^{//} := \{(t,y,\tau,\eta);\ \eta \in \mathit{vect}\,(g\overrightarrow{B})\}. \tag{61}$$

Proof. Set:

$$Ru := \mathrm{curl}\,(\partial_t u \wedge \overrightarrow{B}) = \partial_t \begin{pmatrix} -\partial_{z_2}u_2 - \partial_{z_3}u_3 \\ \partial_{z_1}u_2 \\ \partial_{z_1}u_3 \end{pmatrix}. \tag{62}$$

<u>Transversal measure</u>. The measure μ_T does not charge neither $\mathcal{H}_T$ nor $\mathcal{E}_T$. Thus, it suffices to check:

$$\mathrm{supp}\,\mu_T \mathbb{1}_{(0,T)} \mathbb{1}_{\{x_n>0\}\cup\mathcal{G}_T} \subset \overrightarrow{B}^\perp.$$

Near the boundary, by proposition 7.1 of the appendix,

$$Ru^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } L^2([0,l[, \boldsymbol{H}^{-1}_{\mathrm{loc}}(X')).$$

Thus, according to the decoupling lemma, whether A_0 has support in the interior or near the boundary:

$$\forall A_0 \in \mathcal{A}^0,\ \mathrm{supp}\,A_0 \subset \{t \in (0,T)\},\ (A_0 R u_T^k, u_T^k) \underset{k\to+\infty}{\longrightarrow} 0. \tag{63}$$

Because $\mathrm{div}\,u_T^k = 0$, formula (62) may be written: $Ru_T^k = \partial_t\partial_{z_1}u_T^k$. This implies (by paragraph 3.2.2):

$$\mu_T\Big(\{t \in (0,T)\} \cap \{\sigma_2(\partial_t\partial_{z_1}) \neq 0\}\Big) = 0.$$

This clearly shows the announced result near an interior point. When $\tilde{\rho} \in \mathcal{G}_T$ one may write: $\partial_{z_1} = f_0 \partial_{x_n} + F_1$, where f_0 is a function and F_1 a first order tangential differential operator, and the following basic fact completes the proof:

$$\sigma(F_1)(\tilde{\rho}) = 0 \iff \tilde{\eta}' \perp \overrightarrow{B}.$$

Longitudinal measure. The first coordinate of Ru_L^k is:

$$-\partial_t(\partial_{z_2}^2 + \partial_{z_3}^2)\varphi^k.$$

Its scalar product with $\partial_t \varphi^k$ gives:

$$\forall A_0 \in \mathcal{A}^0,\ \operatorname{supp}(A_0) \subset \{t \in (0,T)\} \Rightarrow \lim_{k \to +\infty} \left(A_0(\partial_{z_2}^2 + \partial_{z_3}^2)\partial_t \varphi^k, \partial_t \varphi^k\right) = 0.$$

This implies (again by paragraph 3.2.2), that μ (thus μ_L) vanishes, in $(0,T)$, on the set of all ρ such that:

$$\rho \in \{x_n > 0\} \cup \mathcal{G}_T, \quad \sigma_2(\partial_{z_2}^2 + \partial_{z_3}^2)(\rho) \neq 0.$$

Hence (61). □

4.3. **Support of the measure near the boundary.** For any symbol q_0 with support in $\{x_n > 0\}$, we have:

$$\left\langle \mu_T, H_{p_T/\tau} q_0 \right\rangle = 0 \qquad \left\langle \mu_L, H_{p_L/\tau} q_0 \right\rangle = 0,$$

which shows, in the interior of Ω, the invariance of each measure by the appropriate hamiltonian flow. Unfortunately, the condition:

$$u^k_{T \restriction \partial\Omega} + u^k_{L \restriction \partial\Omega} = 0 \tag{64}$$

is not always sufficient to extend such a property in the neighbourhood of $\partial\Omega$. Indeed, without any additional assumption, μ_L and μ_T are not deterministic: the value of the two measures for time $t < t_0$ is not uniquely determined by their value for time $t \leq t_0$. In our case, this convenient property holds thanks to the strong conditions on the support of μ_T and μ_L. As announced before, we shall only describe the propagation of the supports of the measure.

Lemma 4.5. *Let μ_T and μ_L be the defect measures associated to a sequence of solutions of the Lamé system satisfying (59). Let:*

$$\tilde{\rho} = (\tilde{x}', \tilde{\xi}') = (\tilde{t}, \tilde{y}', \tilde{\tau}, \tilde{\eta}') \in S^*\partial X$$

and $\mathbf{n}$ the unitary exterior normal vector to $\partial\Omega$ at $\tilde{y}'$. Them μ_T and μ_L both vanish near $\tilde{\rho}$ except possibly in the following cases (cf figure 2):

1. *μ_L is null. The support of μ_T propagates along the transversal flow and:*
   - *$(\mathcal{H}_{T[1]})$ case: $\tilde{\rho} \in \mathcal{H}_T$, $\tilde{\eta}' = 0$ and $\overrightarrow{B}$ is orthogonal to $\mathbf{n}$;*
   - *$(\mathcal{H}_{T[2]})$ case: $\tilde{\rho} \in \mathcal{H}_T$, $\tilde{\eta}' \neq 0$ and $\overrightarrow{B}$ is normal to the reflection plane;*
   - *$(\mathcal{G}_{T[1]})$ case: $\tilde{\rho}$ is diffractive for the transversal wave (i.e. $\tilde{\rho} \in \mathcal{G}_T$ and the bicharacteristic ray passing through $\tilde{\rho}$ only intersect the boundary at $\tilde{\rho}$), and $\overrightarrow{B}$ is orthogonal to $\tilde{\eta}'$;*
   - *$(\mathcal{G}_{T[2]})$ case: $\tilde{\rho} \in \mathcal{G}_T$ is not diffractive for the transversal wave, and $\overrightarrow{B}$ is normal to the reflection plane.*
2. *μ_T is null, the support of μ_L propagates along the longitudinal flow and:*
   - *$(\mathcal{H}_L)$ case $\tilde{\rho} \in \mathcal{H}_L$, $\tilde{\eta}' = 0$ and $\overrightarrow{B}$ is parallel to $\mathbf{n}$;*
   - *$(\mathcal{G}_L)$ case $\tilde{\rho}$ is a diffractive point for the longitudinal wave and $\overrightarrow{B}$ is parallel to $\tilde{\eta}'$.*
3. *Both measures μ_T and μ_L are non null, $\tilde{\rho} \in \mathcal{H}_T \cap \mathcal{H}_L$ and:*
   - *$(T \to L)$ case: $\overrightarrow{B}$ is orthogonal to the transversal ray coming in, and parallel to the longitudinal ray going out of $\tilde{\rho}$. The support of μ_T is an union of incoming transversal rays. The support of μ_L is the union of all outgoing longitudinal rays going out of points of $\mathcal{H}_T \cap \mathcal{H}_L$ where the transversal rays of the support of μ_T come in;*

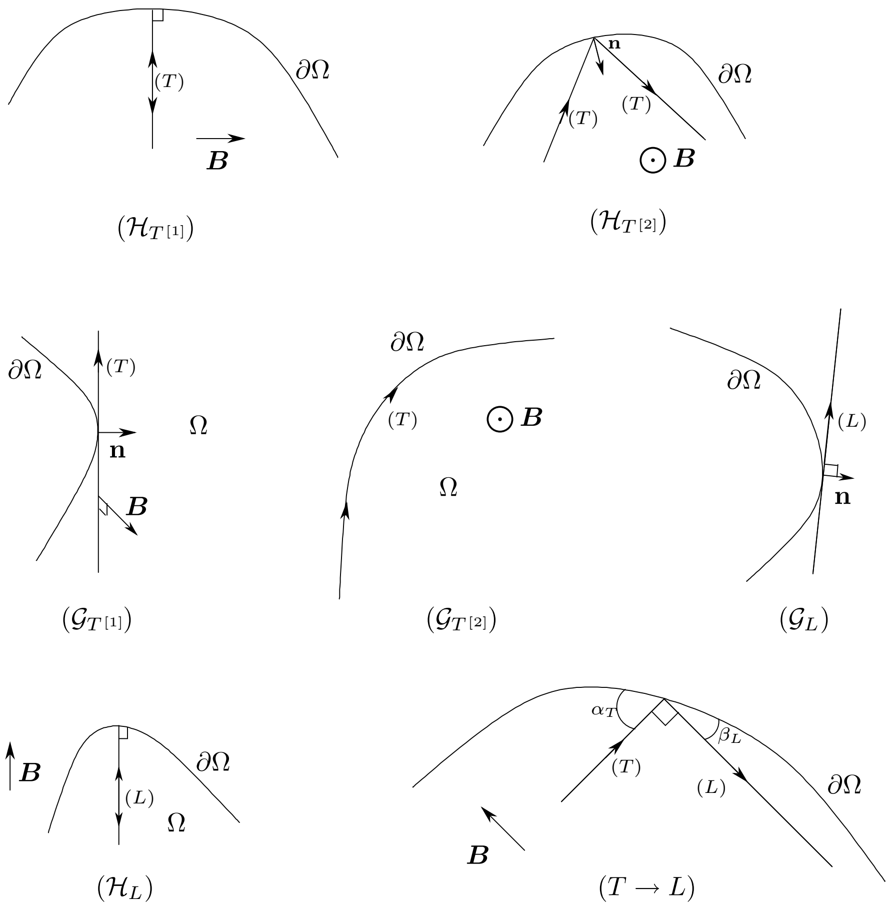

FIGURE 2. Cases arising in lemma 4.5

- <u>$(L \to T)$ case:</u> *$\overrightarrow{B}$ is orthogonal to the transversal ray going out of and parallel to the longitudinal ray coming in $\tilde{\rho}$. The support of μ_L is an union of incoming longitudinal rays. The support of μ_T is the union of all outgoing transversal rays going out of points of $\mathcal{H}_T \cap \mathcal{H}_L$ where the longitudinal rays of the support of μ_L come in.*

All the assertions of lemma 4.5 should be understood in a neighbourhood of $\tilde{\rho}$. The reflection plane at a boundary point $\rho = (t, y', \tau, \eta')$, defined as long as $\eta' \neq 0$, is the plane passing through y' and generated by $\mathbf{n}$ and η', thus containing the bicharacteristic ray passing through ρ. The statement "$\overrightarrow{B}$ is parallel to η'" must be understood as "the vector $g\overrightarrow{B}$ of the cotangent bundle of Ω is parallel to η'".

Notation. Let ρ be an hyperbolic point for the transversal (respectively longitudinal) wave. We shall denote by ξ_T^-, ξ_T^+ (respectively ξ_L^-, ξ_L^+) the incoming and outgoing vectors through ρ:

$$\xi_T^+ := \begin{pmatrix} \xi' \\ \xi_{nT} = \sqrt{\nu_T^2\tau^2 - \|\eta'\|^2} \end{pmatrix} \quad \xi_T^- = \begin{pmatrix} \xi' \\ -\xi_{nT} \end{pmatrix}$$

(respectively with "L" instead of "T").

We shall write $\eta_T^\pm$, $\eta_L^\pm$ the spatial components of this vectors. For example:

$$\eta_T^+ = \begin{pmatrix} \eta' \\ \sqrt{\nu_T^2\tau^2 - \|\eta'\|^2} \end{pmatrix}.$$

Let's postpone the proof of lemma 4.3 to show, as stated in the introduction of this article, that in the $(T \to L)$ and $(L \to T)$ cases, the angles of refraction and incidence have a fixed value, determined by the quotient c_T/c_L. Consider for example the $(T \to L)$ case. Let α_T be the angle of incidence of the transversal wave, β_L the angle of refraction of the longitudinal wave, a_T and a_L the following numbers:

$$a_T := \tan\alpha_T = \frac{\tilde{\xi}_{nT}}{\|\tilde{\eta}'\|}, \quad a_L := \tan\beta_L = \frac{\tilde{\xi}_{nL}}{\|\tilde{\eta}'\|}.$$

The incident and refracted waves are orthogonal, so that:

$$\|\tilde{\eta}'\|^2 - \tilde{\xi}_{nT}\tilde{\xi}_{nL} = 0, \text{ i.e. } a_T a_L = 1. \tag{65}$$

Furthermore, the definition of $\tilde{\xi}_{nL}$ and $\tilde{\xi}_{nT}$ yields

$$c_T^2\|\tilde{\eta}_T^+\|^2 - \tau^2 = 0, \quad c_L^2\|\tilde{\eta}_L^-\|^2 - \tau^2 = 0,$$

which gives the equation:

$$c_T^2(1 + a_T^2) = c_L^2(1 + a_L^2). \tag{66}$$

Equations (65) and (66) imply the formula announced in the introduction:

$$\alpha_T = \arctan\frac{c_L}{c_T}, \ \beta_L = \arctan\frac{c_T}{c_L}.$$

By a similar calculation, one gets, in the $(T \to L)$ case:

$$\alpha_L = \arctan\frac{c_T}{c_L}, \ \beta_T = \arctan\frac{c_L}{c_T}.$$

There are thus very strong constraints for the possible transfer of energy from one wave equation to the other.

Proof of lemma 4.5. <u>Case (3)</u>: $\mu_T \neq 0$, $\mu_L \neq 0$ near $\tilde{\rho}$.

In this case, $\tilde{\rho} \notin \overline{\mathcal{E}_T \cup \mathcal{E}_L}$. It is also easy to show that $\tilde{\rho} \notin \mathcal{G}_T \cup \mathcal{G}_L$. Indeed, if $\tilde{\rho} \in \mathcal{G}_T$ then it also belongs to $\mathcal{H}_L$ (it cannot be a point of $\mathcal{E}_L$, and $\mathcal{G}_L$ and $\mathcal{G}_T$ are disjoint). But μ_T being non-null near $\tilde{\rho}$, $\tilde{\eta}$ is orthogonal to $\overrightarrow{B}$ (by lemma 4.4) so neither $\tilde{\eta}_L^+$ nor $\tilde{\eta}_L^-$ are parallel to $\overrightarrow{B}$, which implies (again by lemma 4.4) that $\mu_L = 0$ near $\tilde{\rho}$, contradicting our assumptions. Likewise, if $\tilde{\rho} \in \mathcal{G}_L$, $\tilde{\eta}'$ must be parallel to $\overrightarrow{B}$ and μ_T null near $\tilde{\rho}$. Thus $\tilde{\rho} \in \mathcal{H}_T \cap \mathcal{H}_L$. The support of measures μ_T and μ_L is, near $\tilde{\rho}$, an union of incoming and outgoing maximal rays.

Let's first assume that the support of μ_L contains the ray going out of $\tilde{\rho}$. Then:

$$\tilde{\eta}_L^+ // \overrightarrow{B}$$

so that $\tilde{\eta}_T^+$ is not orthogonal to $\overrightarrow{B}$. As a consequence, the support of μ_T is only made of incoming rays, and the fact that $\mu_T \neq 0$ implies:

$$\tilde{\eta}_T^- \perp \overrightarrow{B}.$$

Thus $\tilde{\eta}_L^-$ is not parallel to $\overrightarrow{B}$. This is the $(T \to L)$ case, and it remains to show the statement of the lemma about the transfer from transversal incoming waves to longitudinal outgoing waves, which may be formulated as follow: for any $\dot{\rho} \in \mathcal{H}_T \cap \mathcal{H}_L$ near $\tilde{\rho}$ the following equivalence holds:

$$\dot{\rho} \in \operatorname{supp}\mu_T \iff \dot{\rho} \in \operatorname{supp}\mu_L. \tag{67}$$

Let's assume for example $\dot{\rho} \notin \operatorname{supp} \mu_T$. Then μ_T is null near rays coming in and going out of $\dot{\rho}$ and by the hyperbolic theory (see proposition 3.16):

$$u^k_{L\restriction x_n=0} = -u^k_{T\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\dot{\rho},\partial}.$$

By the propagation theorem, the support of μ_L propagates near $\dot{\rho}$. But this support is an union of outgoing rays. Consequently, it is empty near $\dot{\rho}$ and μ_L is null near $\dot{\rho}$. The implication $\dot{\rho} \notin \operatorname{supp} \mu_L \Rightarrow \dot{\rho} \notin \operatorname{supp} \mu_T$ may be shown in the same manner.

If the support of μ_L contains no outgoing ray near $\tilde{\rho}$, it must contain incoming rays. This corresponds to the $(L \to T)$ case, which may be treated as the $(T \to L)$ case.

The study of all other cases relies on a transfer argument on boundary conditions, stated in the following technical lemma 4.6: rougly, a boundary condition on the longitudinal wave implies one on the transversal wave and vice versa.

Notation. Let $v^k_{T,L}$ be the functions $u^k_{T,L}$ considered as vector fields on Ω. In local coordinates, if χ denotes the change of coordinates, we have:

$$v^k_T = \chi'(y) u^k_T = \begin{pmatrix} v^k_{T1} \\ v^k_{T2} \\ v^k_{Tn} \end{pmatrix} \qquad v^k_L = \chi'(y) u^k_L = \begin{pmatrix} v^k_{L1} \\ v^k_{L2} \\ v^k_{Ln} \end{pmatrix}.$$

Lemma 4.6. *Let $\tilde{\rho} \in S^*\partial X$, and (u^k) be any sequence of solutions of the Lamé system weakly converging to* 0 *in $\boldsymbol{H}^1_{\text{loc}}(\mathbb{R} \times \overline{\Omega})$.*

- *Assume the following aproximate equation for some $A_1 \in \mathcal{A}^1$:*

(68)
$$\partial_{x_n} v^k_{Tn\restriction x_n=0} = A_1 v^k_{Tn\restriction x_n=0} + o(1) \text{ in } L^2_{\tilde{\rho},\partial}.$$

  Then:

(68')
$$\Delta_{y'} \varphi^k_{\restriction x_n=0} = -A_1 \partial_{x_n} \varphi^k_{\restriction x_n=0} + o(1) \text{ in } H^2_{\tilde{\rho},\partial}.$$

- *Conversely, if, for some $A_{-1} \in \mathcal{A}^{-1}$ the following equation holds:*

(69)
$$\varphi^k_{\restriction x_n=0} = A_{-1} \partial_{x_n} \varphi^k_{\restriction x_n=0} + o(1) \text{ in } H^2_{\tilde{\rho},\partial}.$$

  Then:

(69')
$$\partial_{x_n} v^k_{Tn\restriction x_n=0} = -\Delta_{y'} A_{-1} v^k_{Tn\restriction x_n=0} + o(1) \text{ in } L^2_{\tilde{\rho},\partial}.$$

- *Moreover if, in addition to (69), $\tilde{\eta}' \neq 0$ and $\sigma(A_{-1})(\tilde{\rho}) \neq 0$, then:*

(69")
$$v^k_{T\restriction x_n=0} = \mathbf{Z}_{-1} \partial_{x_n} v^k_{T\restriction x_n=0} + o(1) \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}$$

$$\mathbf{Z}_{-1} \in \boldsymbol{\mathcal{A}}^{-1},\ \sigma_{-1}(\mathbf{Z}_{-1}) = \|\eta'\|^{-2} \begin{pmatrix} 0 & 0 & ig'^{-1} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \\ 0 & 0 & \\ 0 & 0 & \sigma(A_{-1})^{-1} \end{pmatrix}.$$

Proof. First note that the Dirichlet condition on u^k implies:

(70)
$$\partial_{x_n} v^k_{Tn\restriction x_n=0} = \Delta_{y'} \varphi^k_{\restriction x_n=0} + O(1) \text{ in } H^{1/2}_{\text{loc}}(\partial X),$$

Indeed, the equation $\operatorname{div} u^k_T = 0$ implies, in local coordinates:

$$\partial_{x_n} v^k_{Tn} + \partial_{y_1} v^k_{T1} + \partial_{y_2} v^k_{T2} = O(1) \text{ in } H^1_{\text{loc}}(X).$$

Which may be written, using $v^k_{n\restriction x_n=0} = 0$:

$$\partial_{x_n} v^k_{Tn\restriction x_n=0} = \partial_{y_1} v^k_{L1\restriction x_n=0} + \partial_{y_2} v^k_{L2\restriction x_n=0} + O(1) \text{ in } H^{1/2}_{\text{loc}}(\partial X),$$

yielding (70) by the definition of φ.

Assume (68). By (70) and the nullity of $v^k_{n\restriction x_n=0}$:

$$\Delta_{y'}\varphi^k_{\restriction x_n=0} = -A_1 v^k_{Ln\restriction x_n=0} + o(1) = -A_1\partial_{x_n}\varphi^k_{\restriction x_n=0} + o(1) \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}.$$

Now assume (69). Hence:

$$\Delta_{y'}\varphi^k_{\restriction x_n=0} = \Delta_{y'}A_{-1}\partial_{x_n}\varphi^k_{\restriction x_n=0} + o(1) \text{ in } L^2_{\tilde\rho,\partial}.$$

Which implies (69') using (70) on the left side of the equation, and the Dirichlet condition on v^k_m on its right side.

If, in addition to the assumption (69), $\tilde\eta'$ and $\sigma(A_{-1})(\tilde\rho')$ are non zero, both operators A_{-1} and $\Delta_{y'}$ are elliptic at $\tilde\rho$, and equations (69') and (70) may be rewritten:

$$v^k_{Tn\restriction x_n=0} = Y_{-1}\partial_{x_n}v^k_{Tn\restriction x_n=0} + o(1) \text{ in } L^2_{\tilde\rho,\partial} \quad \sigma(Y_{-1}) = \|\tilde\eta'\|\|^{-2}\sigma(A_{-1})^{-1}, \tag{71}$$
$$\varphi^k_{\restriction x_n=0} = E_{-2}\partial_{x_n}v^k_{Tn\restriction x_n=0} + O(1) \text{ in } H^{3/2}_{\tilde\rho,\partial}, \quad \sigma(E_{-2}) = -\|\eta'\|^{-2} \text{ near } \tilde\rho$$

$$\begin{pmatrix} v^k_{T1\restriction x_n=0} \\ v^k_{T2\restriction x_n=0}\end{pmatrix} = g'^{-1}\begin{pmatrix} \partial_{y_1}\varphi^k_{\restriction x_n=0} \\ \partial_{y_2}\varphi^k_{\restriction x_n=0}\end{pmatrix} = Z_{-1}\partial_{x_n}v^k_{Tn\restriction x_n=0} + O(1) \text{ in } H^{1/2}_{\tilde\rho,x_n=0} \tag{72}$$

$$\sigma(Z_{-1}) = -i\|\eta'\|^{-2}g'^{-1}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \text{ near } \tilde\rho.$$

Assertion (69") is an easy consequence of (71) and (72). □

We may now study cases (1) and (2) of lemma 4.5.

case (1): assume $\mu_L = 0$, $\mu_T \neq 0$ near $\tilde\rho$. There are three possibilities:

- If $\tilde\rho \in \mathcal{H}_L$, the nullity of μ_L implies, by standard hyperbolic theory:

  $$u^k_{T\restriction x_n=0} = -u^k_{L\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}.$$

  So near $\tilde\rho$, the support of μ_T propagates. It is easy to see that condition (60) on the support of μ_T implies, if μ_T does not vanish, that this is one of the four cases described in the (1) of lemma 4.5.
- Assume $\tilde\rho \in \mathcal{E}_L$. The standard elliptic theory (proposition 3.17) implies:

  $$\partial_{x_n}\varphi^k_{\restriction x_n=0} + \Xi\varphi^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H^1_{\tilde\rho,\restriction x_n=0}.$$

  With lemma 4.6, this yields the following equation on the traces of u^k_T:

  $$u^k_{T\restriction x_n=0} = i\widetilde{\mathbf{Z}}_{-1}D_{x_n}u^k_{T\restriction x_n=0} + o(1) \text{ in } \boldsymbol{H}^1_{\tilde\rho,\partial}, \quad \widetilde{\mathbf{Z}}_{-1} = \chi'^{-1}\mathbf{Z}_{-1}\chi' \tag{73}$$

  where the principal symbol of the operator $\mathbf{Z}_{-1} \in \boldsymbol{\mathcal{A}}^{-1}$ is given by (69"). Notice that the eigenvalues of $\sigma_{-1}(i\mathbf{Z}_{-1})$, thus those of $\sigma_{-1}(i\widetilde{\mathbf{Z}}_{-1})$ are pure imaginary numbers. As a consequence, the boundary condition (73) is an uniform Lopatinsky boundary condition near $\tilde\rho$ (see the example following definition 3.11), which shows again the propagation of μ_T. As in the case where $\tilde\rho \in \mathcal{H}_L$, it is easy to see that this is one of the four cases of lemma 4.5.
- The case $\tilde\rho \in \mathcal{G}_L$ is the most difficult. When μ_T is non-null $\tilde\rho$, must be in $\mathcal{H}_T$. We use a contradiction argument to prove the propagation of the support of μ_T. Let $\dot\rho \in \mathcal{H}_T$ such that the ray coming in $\dot\rho$ is in the support of μ_T, but not the ray going out of $\dot\rho$. According to the standard hyperbolic theory (proposition 3.16):

  $$D_{x_n}u^k_{T\restriction x_n=0} - \Lambda_T u^k_{T\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde\rho,\partial}, \quad \sigma_1(\Lambda_T) = \sqrt{\nu_T\tau^2 - \|\eta'\|^2}.$$

  This implies, by lemma 4.6, a boundary equation on $\partial_t\varphi^k$, of the following form:

  $$\partial_t\varphi^k_{\restriction x_n=0} = Y_{-1}D_{x_n}\partial_t\varphi^k_{\restriction x_n=0} + o(1) \text{ in } H^1_{\dot\rho,\partial},$$

which is an uniform Lopatinsky condition near $\dot{\rho}$ because $\tilde{\rho} \in \mathcal{G}_L$. In view of proposition 3.19 on traces in the glancing region, such an equation implies, with the nullity of μ near $\dot{\rho}$ the following conditions:

$$\partial_t \varphi^k_{\restriction x_n=0} \to 0 \text{ in } H^1_{\dot{\rho},\partial}, \quad \partial_n \partial_t \varphi^k_{\restriction x_n=0} \to 0 \text{ in } L^2_{\dot{\rho},\partial}.$$

The operator ∂_t being elliptic at $\dot{\rho}$, this shows that u^k_L tends to 0 in $\boldsymbol{H}^1_{\dot{\rho},\partial}$, and thus:

$$u^k_{T\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\dot{\rho},\partial}.$$

Hence the propagation of the support of μ_T near $\dot{\rho}$, which contredicts the assumption on rays coming in and going out of $\dot{\rho}$.

Similar arguments show that if the ray going out of $\dot{\rho}$ is in the support of μ_T, so is the ray coming in $\dot{\rho}$. This proves that the support of μ_T propagates near $\tilde{\rho}$. Notice that this is necessarily the $(\mathcal{H}_{T[2]})$ case.

Remark 4.7. Case (1), which appears in the study of linear thermoelasticity, was precisely described in [2]. The authors show a result of propagation of μ_T, determining all the characteristic elements of this propagation, which gives in particular the polarization properties of μ_T. In the case of the system of magnetoelasticity, the polarization causes no problem by and it suffices to show the propagation of the support of μ_T (or that of μ_L in case (2)). As mentionned in the introduction, one may consider that the only component of μ_T and μ_L which is resistant to the dissipation is the component parallel to $\boldsymbol{B}$, cancelling the quantity $u \wedge \boldsymbol{B}$.

Case 2: we assume now that $\mu_T = 0$ and $\mu_L \neq 0$. We argue in a similar way, considering three possibilities:

- If $\tilde{\rho} \in \mathcal{H}_T$, the standard hyperbolic theory gives an approximate boundary equation on u^k_T, which implies:

$$u^k_{L\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial}.$$

  As a consequence, the support of μ_L propagates. On this support, η is parallel to $\overrightarrow{B}$, which shows, as stated in lemma 4.5, that $\tilde{\eta}' = 0$ and $\tilde{\rho} \in \mathcal{H}_L$, or $\tilde{\eta}' // \overrightarrow{B}$ and $\tilde{\rho} \in \mathcal{G}_L$.
- If $\tilde{\rho} \in \mathcal{E}_T$, we write (as in the similar situation when $\tilde{\rho} \in \mathcal{E}_L$), the boundary equation of the elliptic region:

$$D_{x_n} u^k_{T\restriction x_n=0} + \Xi_T u^k_{T\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial}.$$

  This implies in view of lemma 4.6 an uniform Lopatinsky boundary equation on $\partial_t \varphi^k$, thus the propagation of the support of μ, which is the same as that of μ_L. The fact that $\eta' \neq 0$ shows that $\tilde{\rho}$ cannot be hyperbolic for the longitudinal wave (in this case outgoing and incoming directions are not parallel, thus at least one is not parallel to $\overrightarrow{B}$). Consequently, $\tilde{\rho} \in \mathcal{G}_L$. More precisely, it is a diffractive point: the bicharacteristic passing through $\tilde{\rho}$ must stay parallel to $\overrightarrow{B}$, thus its direction is constant which is not possible for gliding rays because Ω has no contact of infinite order with its tangents. We are in the $(\mathcal{G}_L)$ case of lemma 4.5.
- If $\tilde{\rho} \in \mathcal{G}_T$, then $\tilde{\rho} \in \mathcal{H}_L$. The fact that $\tilde{\rho} \in \mathcal{G}_T$ implies that $\tilde{\eta}' \neq 0$, so directions $\tilde{\eta}^+_L$ and $\tilde{\eta}^-_L$ cannot be both parallel to $\overrightarrow{B}$. Consequently, the support of μ_L is an union of only ingoing rays (or only outgoing rays). This gives a boundary equation of the following form:

$$\partial_{x_n} \partial_t \varphi^k_{\restriction x_n=0} + \iota \Lambda_L \partial_t \varphi^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}^2_{\tilde{\rho},\partial},$$

  where $\iota \in \{+1, -1\}$. Notice that ∂_t is elliptic at $\tilde{\rho}$, so that we may rewrite this last property taking out all the ∂_t and with $\boldsymbol{H}^1$ instead of $\boldsymbol{L}^2$. This yields, in view of lemma 4.6, a unform Lopatinsky boundary condition on u^k_T. The nullity of μ_T gives as before (by proposition 3.19):

$$u^k_L = -u^k_T \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\tilde{\rho}},$$

so that μ_L propagates, and in view of the particular form of its support, vanishes near $\tilde{\rho}$. This shows that this particular situation ($\tilde{\rho} \in \mathcal{G}_T$ and $\mu_L \neq 0$) is impossible, and completes the proof of lemma 4.5.

□

Definition 4.8. We shall call $\boldsymbol{B}$**-admissible** points the point of the boundary of $S_b^* M$ which are of one of the eight types described in lemma 4.5.

4.4. **Conclusion of the proof.** Let $S := \operatorname{supp} \mu_L \cup \operatorname{supp} \mu_T$, and BR the subset of $S_b^* M$, of all points ρ satisfying one of the following properties:

- $x_n > 0$, $\rho \in S\widehat{Z}_T$ and $\eta // \overrightarrow{B}$;
- $x_n > 0$, $\rho \in S\widehat{Z}_L$ and $\eta \perp \overrightarrow{B}$;
- $x_n = 0$ and ρ is $\boldsymbol{B}$-admissible.

Let $\Phi_T(\rho, s)$ and $\Phi_L(\rho, s)$ be the bicharacteristic flows for the transversal and longitudinal waves. We shall define a local continuous flow on BR, denoted by:

$$\Phi(\rho, s) = (\Phi_{x'}, \Phi_{x_n}, \Phi_{\xi'}, \Phi_{\xi_n}),$$

in the following way:

- if $\Phi_{x_n}(\rho, s) > 0$ and $\Phi(\rho, s) \in S\widehat{Z}_T$, or if $\Phi(\rho, s)$ is a $\boldsymbol{B}$-admissible boundary point of the form (1) of lemma 4.5, Φ is near (ρ, s) the restriction to BR of the transversal bicharacteristic flow;
- if $\Phi_{x_n}(\rho, s) > 0$ and $\Phi(\rho, s) \in S\widehat{Z}_L$, or if $\Phi(\rho, s)$ is a $\boldsymbol{B}$-admissible boundary point of the form (2) of lemma 4.5, Φ is near (ρ, s) the restriction to BR of the longitudinal bicharacteristic flow;
- if $\Phi(\rho, s)$ is $\boldsymbol{B}$-admissible of type $(T \to L)$, then:

$$\Phi(\rho, r) = \Phi_T(\rho, r), \text{ if } r < s$$
$$\Phi(\rho, r) = \Phi_L(\rho, r), \text{ if } r > s;$$

- if $\Phi(\rho, s)$ is $\boldsymbol{B}$-admissible of type $(L \to T)$, then:

$$\Phi(\rho, r) = \Phi_L(\rho, r), \text{ if } r < s$$
$$\Phi(\rho, r) = \Phi_T(\rho, r), \text{ if } r > s.$$

In view of lemma 4.3, S (which a subset of BR) is stable under the flow Φ on $(0, T)$. Furthemore, if for some $\rho \in \mathsf{BR}$,

$$\Phi(\rho, s) \underset{s \to \tilde{s}}{\longrightarrow} \tilde{\rho} \notin \mathsf{BR}, \ \tilde{s} \in (0, T),$$

(thus $\tilde{\rho}$ is a boundary point which is not $\boldsymbol{B}$-admissible), then ρ is not in S. Consequently, S is an union of $\boldsymbol{B}$-resistant rays of life-length T. The assumption of non-existence of such rays made in proposition 4.1 shows that S is empty, which completes the proof.

5. Necessary condition

Proposition 5.1. *Assume that for all $T > 0$, there exists a $\boldsymbol{B}$-resistant ray of life-length T. Then for all $T > 0$, there exists a sequence (u^k) of solutions of the Lamé system such that:*

$$\|\partial_t u^k_{\restriction t=0}\|^2_{\boldsymbol{L}^2} + \|u^k_{\restriction t=0}\|^2_{\boldsymbol{H}^1_0} \underset{k \to +\infty}{\longrightarrow} 1 \tag{74}$$

$$\|u^k \wedge \boldsymbol{B}\|_{\boldsymbol{H}^1((0,T)\times\Omega)} \underset{k \to +\infty}{\longrightarrow} 0 \tag{75}$$

Corollary 5.2. *Under the assumption of proposition 5.1, the energy of the solutions of the magnetoelasticity equations does not decay uniformly. In other terms, the necessary condition of theorem 1 holds.*

Corollary 5.2 is a direct consequence of point b) of proposition 2.5.

Proof of proposition 5.1. This proof is very much inspired by that of theorem 4 of [2]. Denote by Z one of the indices T or L and set $P_Z := \Delta - \nu_Z^2 \partial_t^2$. We start by an elementary remark:

Remark 5.3. If E is a vector subspace of $\mathbb{C}^3$ and π_E the orthogonal projection on E, the defect measure of $\pi_E u_Z^k$ is $\pi_E^* \mu_Z \pi_E$. Furthermore, $P_Z \pi_E u_Z^k = 0$, so that theorem of propagation 3.13 holds for the measure $\pi_E^* \mu_Z \pi_E$ if an uniform Lopatinsky boundary condition holds on $\pi_E u_Z^k$. Notice that any **scalar** uniform Lopatinsky condition on u_Z^k yields such a condition on $\pi_E u_Z^k$. If $\pi_E^* \mu_Z \pi_E = \mu_Z$, the measure μ_Z will be said to be **polarized along** E. If E is the line generated by a vector H of $\mathbb{C}^3$ we shall also use the phrase "polarized along H". If both measures $\mu_T 1\!\!1_{]-\varepsilon, T+\varepsilon[}$ and $\mu_L 1\!\!1_{]-\varepsilon, T+\varepsilon[}$ are polarized along $\boldsymbol{B}$, then condition (75) is fullfilled.

Let $T' > T$. Consider a $\boldsymbol{B}$-resistant ray defined on an open interval I of length T':

$$\gamma(s) = (t_\gamma(s), y_\gamma(s), \tau_\gamma(s), \eta_\gamma(s)) = (x_\gamma(s), \xi_\gamma(s))$$

If T' is large enough, then one of the two following assertions holds:

a) $\gamma(I)$ contains an interior point;
b) $y_\gamma(I) = \Gamma \subset \partial\Omega$ where Γ is a closed curved, contained in a plane P which is normal to $\boldsymbol{B}$, boundary of a convex subset of P, and such that on Γ, $\mathbf{n}$ is orthogonal to $\boldsymbol{B}$.

Case b) occures when there exists an infinite boundary $\boldsymbol{B}$-resistant ray. This case reduces to case a), choosing a transversal ray contained in P which only meets the boundary at hyperbolic points.

Thus, we may assume that $\gamma(I)$ has an interior point. We may also assume, possibly moving the origin of coordinates, that this interior point is $\gamma(0)$, and that $(t_\gamma(0), y_\gamma(0)) = (0, 0)$. Recall that the magnetic field is vertical: $\boldsymbol{B} = (B, 0, 0)$. We shall denote by $-T^-$ and T^+ the extremal points of I: $I = (-T^-, T^+)$.

If $\eta_\gamma(0)$ is parallel to $\boldsymbol{B}$ (i.e. if $\gamma(0)$ is in the longitudinal characteristic set), choose a non-zero function $\varphi \in C_0^\infty(\Omega)$, and set:

$$\varphi^k(y) = K^{-1} k^{-5/4} e^{iky_1} \varphi(\sqrt{k} y)$$
$$u_0^k = \nabla \varphi^k, \quad u_1^k = ikc_L u_0^k.$$

Where u^k is the solution of the Lamé system with initial data:

$$(u^k, \partial_t u^k)_{\restriction t=0} = (u_0^k, u_1^k).$$

Then:

$$\|u_0^k\|_{\boldsymbol{H}^1} \underset{k\to+\infty}{\longrightarrow} K^{-1} \|\varphi\|_{\boldsymbol{L}^2}, \quad \|u_1^k\|_{\boldsymbol{L}^2} \underset{k\to+\infty}{\longrightarrow} K^{-1} \|\varphi\|_{\boldsymbol{L}^2}.$$

Thus, condition (74) is fullfilled with an appropriate choice of K.

For small t, by finite speed of propagation for the wave equation, u_L^k has compact support in Ω. Thus $u_T^k = 0$ and $u_L^k = u^k$. As a consequence, for small t:

(1) $\mu_T = 0$;
(2) μ_L is polarized along $\boldsymbol{B}$;
(3) the projection of the support of μ_L on $\mathbb{R}_t \times \overline{\Omega}$ is contained in $x_\gamma(I)$.

If $\eta_\gamma(0)$ is orthogonal to $\boldsymbol{B}$, we construct a sequence of solution of the Lamé system, with the following initial data: (cf [2])

$$\psi^k = K^{-1} k^{-5/4} e^{iky_1} \psi(\sqrt{k} y)$$
$$u_0^k = \operatorname{curl}(0, 0, -\psi^k), \quad u_1^k = ikc_T u_0^k.$$

In this case, condition (74) is fullfilled for an appropriate K and the defect measures satisfy the following properties for small t:

(1) $\mu_L = 0$;
(2) μ_T is polarized along $\boldsymbol{B}$;
(3) the projection of the support of μ_T on $\mathbb{R}_t \times \overline{\Omega}$ is contained in $x_\gamma(I)$.

To show (75), we shall prove that both measures μ_T and μ_L, are, for $t \in I$, polarized along $\boldsymbol{B}$. For $t > 0$, we shall denote by $\mathcal{P}(t)$ the following property: **in a neighbourhood of** $[0,t]$**, both measures** μ_L **and** μ_T **are polarized along** $\boldsymbol{B}$ **and the projections of their support on** $\mathbb{R}_t \times \overline{\Omega}$ **are contained in** $x_\gamma(I)$.
Let $\mathcal{T}$ be the set of t in $[0, T^+)$ such that $\mathcal{P}(t)$ holds. By its definition, $\mathcal{T}$ is an open subset of $[0, T^+)$. We have just shown that 0 in $\mathcal{T}$. We shall now prove that $\mathcal{T}$ is closed. We shall use the next lemma:

Lemma 5.4. *Let* $\tilde{\rho} = (\tilde{t}, \tilde{y}, \tilde{\tau}, \tilde{\eta}) \in S_b^* M$*. If* $\mu_T \mathbb{1}_{t<\tilde{t}}$ *and* $\mu_L \mathbb{1}_{t<\tilde{t}}$ *vanish in a neighbourhood of* $\tilde{\rho}$*, so do both measures* μ_T *and* μ_L*.*

This is a trivial assertion in the interior of Ω by the propagation of both measures. Near a point of the boundary of Ω, one may show lemma 5.4 using the Dirichlet boundary condition on u^k and the theorem of propagation 3.13, together with the same type of arguments as in lemma 4.5.

Let $s_0 > 0$ such that $\mathcal{P}(s_0)$ holds for $s < s_0$. We must check that $\mathcal{P}(s_0)$ holds. Three cases arise, depending on the nature of $\rho := \gamma(s_0)$.

i) ρ is an interior point.

$\mathcal{P}(s_0)$ is obvious in view of the propagation of both measures in the interior of Ω.

ii) ρ is of the type (1) of lemma 4.5.

This case, where μ_L vanishes for time $t < t_\gamma(s_0)$ near $t_\gamma(s_0)$, were studied in [2]. The authors show that μ_L remains null for times greater than $t_\gamma(s_0)$ and that μ_T propagates near $\gamma(s_0)$, in such a way that in our case, its polarization along $\boldsymbol{B}$ is preserved. In particular property $\mathcal{P}(s_0)$ holds.

iii) ρ is of the type $(\mathcal{H}_L)$ of lemma 4.5: $y'_\gamma(s_0) \in \partial\Omega$, $\eta'_\gamma(s_0) = 0$ and $\mathbf{n} // \boldsymbol{B}$.

In view of lemma 5.4, the support of the measure μ_L is contained, near ρ in the union of the longitudinal ray coming in ρ and the ray going out of ρ. The support of μ_T, if not empty, if the transversal ray going out of ρ. Let E be the plane orthogonal to $\boldsymbol{B}$ in $\mathbb{C}^3$. The polarization of μ_L along $\boldsymbol{B}$ shows that $\pi_E^* \mu_L \pi_E = 0$ and thus, by remark 5.3 and the standard hyperbolic theory of proposition 3.15:

$$\pi_E u^k_{T \restriction x_n = 0} = -\pi_E u^k_{L \restriction x_n=0} \rightarrow 0 \text{ in } \boldsymbol{H}^1_{\rho,\partial}$$

which implies, using again remark 5.3 that $\pi_E^* \mu_T \pi_E$ propagates along the transversal flow near $\gamma(s_0)$. Thus $\pi_E^* \mu_T \pi_E$ vanishes near ρ. But μ_T is polarized orthogonally to its direction of propagation which is exactly $\boldsymbol{B}$ on the support of μ_T near ρ. This show that μ_T vanishes near ρ, completing the proof of $\mathcal{P}(s_0)$.

iv) ρ is of the type $(\mathcal{G}_L)$ of lemma 4.5: ρ is a diffractive point for the longitudinal wave, and $\eta'_\gamma(s_0)$ is parallel to $\boldsymbol{B}$.

Then $\rho \in \mathcal{H}_T \cup \mathcal{E}_T$. Furthermore, in the longitudinal hyperbolic case, $\mu_T \mathbb{1}_{t<t_\gamma(s_0)}$ vanishes near ρ. Thus, according to standard elliptic or hyperbolic theory (cf propositions 3.16 and 3.17), u_T^k satisfies a boundary condition of the following form:

$$D_{x_n} u^k_{T \restriction x_n=0} = A u^k_{T\restriction x_n = 0} + o(1) \text{ in } \boldsymbol{L}^2_{\rho,\partial}, \quad A \in \boldsymbol{\mathcal{A}}^1_\partial$$

$$\sigma_1(A) = -i\sqrt{\|\eta'\|^2 - \nu_T^2 \tau^2} \text{ in the elliptic case,}$$

$$\sigma_1(A) = -\sqrt{\nu_T^2 \tau^2 - \|\eta'\|^2} \text{ in the hyperbolic case.}$$

Each of this equation yields, in view of lemma 4.6, an uniform Lopatinsky boundary equation on φ^k:

$$\varphi^k_{\restriction x_n=0} = B_{-1} D_{x_n} \varphi^k_{\restriction x_n=0} + o(1) \text{ in } \boldsymbol{H}^1_{\rho,\partial}. \tag{76}$$

As a consequence, the support of μ (and that of μ_L) propagates near ρ. The polarization of μ_L along $\boldsymbol{B}$ is immediate. The nullity of μ_T near ρ remains to be checked. This is a general property in the elliptic case $\rho \in \mathcal{E}_T$. In the hyperbolic case, first note that the propagation theorem of Burq and Lebeau [2, th. 1] implies with boundary condition (76) that μ is invariant by the longitudinal flow near diffractive points. So the total mass of μ_L is preserved by time, for t close enough to $t_\gamma(s_0)$. The next lemma, which is a measure version of the conservation of energy for the Lamé system, completes the proof of $\mathcal{P}(s_0)$:

Lemma 5.5. *Let $\varphi \in C_0^\infty(\mathbb{R})$. Then:*

$$\left\langle \mu_T + \mu_L, \varphi'(t) \right\rangle = 0.$$

In other terms, the total mass of the measure $(\mu_T + \mu_L)_{\restriction t=s}$ is well defined, and does not depend on s.

Proof.

$$(\partial_t^2 - \Delta_e)u^k = 0$$

$$\operatorname{Re} \int \partial_t^2 u^k \overline{u}^k \varphi(t) dx - \operatorname{Re} \int \Delta_e u^k \overline{u}^k \varphi(t) dx = 0.$$

Set: $\nabla_e u := (\mu \nabla u, (\lambda + \mu) \operatorname{div} u) \in \mathbb{C}^4$. A simple integration by parts yields:

$$\int \varphi'(t) \left|\partial_t u^k\right|^2 dx + \int \varphi'(t) \left|\nabla_e u^k\right|^2 dx.$$

Using another integration by parts, and then the decoupling lemma 3.30:

$$-\int \partial_t^2 u^k \overline{u}^k \varphi'(t) dx - \int \Delta_e u^k \overline{u}^k \varphi'(t) dx = o(1) \text{ as } k \to +\infty$$

$$\int (\partial_t^2 + c_T^2 \Delta) u_T^k \overline{u}_T^k \varphi'(t) dx + \int (\partial_t^2 + c_T^2 \Delta) u_L^k \overline{u}_L^k \varphi'(t) dx = 0.$$

When k tends to ∞, we get, by the definition of μ_T and μ_L:

$$\left\langle \mu_T, \frac{\tau^2 + c_T^2 \|\eta\|^2}{2\tau^2} \varphi'(t) \right\rangle + \left\langle \mu_L, \frac{\tau^2 + c_L^2 \|\eta\|^2}{2\tau^2} \varphi'(t) \right\rangle = 0.$$

This completes the proof, noting that on the support of μ_T (respectively μ_L), $c_T \|\eta\|$ (respectively $c_L \|\eta\|$) is equal to τ. $\square$

Lemma 5.5 and the mass conservation of μ_L as time goes by imply that the mass of μ_T is also preserved near ρ, which shows that μ_T vanishes in a neighbourhood of ρ.

v) ρ is of the type $(L \to T)$ of lemma 4.5.

In view of lemma 5.4 and of the assumption $P(s)$ for $s < s_0$, the support of μ_L is contained in the two longitudinal half-rays passing through ρ, and that of μ_T is only contained in the ray going out of ρ. To prove $P(s_0)$, it remains to show that $\mu_L = 0$ along the longitudinal ray going out of ρ. We shall do so by a simple polarization argument. Let H (respectively J) be an unitary vector of $\mathbb{C}^3$ parallel to the direction of the transversal (respectively longitudinal) ray going out of ρ. The polarization of μ_T shows that $\pi_H^* \mu_T \pi_H$ is null, so that $\pi_H^* \mu_L \pi_H$ propagates near ρ. Now, H is orthogonal to the direction of the longitudinal ray coming in ρ, so that $\pi_H^* \mu_L \pi_H$ vanishes, along incoming rays but also, in view of the propagation, along outgoing rays. Furthermore $\mu_L 1\!\!1_{t > t_\gamma(s_0)}$ is polarized along J. It is easy to see that this last measure vanishes. Indeed, the polarization of μ_L implies:

$$1\!\!1_{t > t_\gamma(s_0)} \pi_J^* \pi_H^* \pi_J^* \mu_L \pi_J \pi_H \pi_J = 1\!\!1_{t > t_\gamma(s_0)} \pi_J^* \pi_H^* \mu_L \pi_H \pi_J = 0$$

But:

$$\pi_J \pi_H \pi_J = \frac{< H, J >^2}{|H|^2 |J|^2} \pi_J$$

Noting that H and J are not orthogonal this yields the nullity of $\mu_L 1\!\!1_{t > t_\gamma(s_0)}$ in a neighbourhood of ρ.

vi) ρ is of the type $(T \to L)$ of lemma 4.5 .

One may argue as before, showing that for every vector K orthogonal to J, $\pi_K^* \mu_T \pi_K = 0$ near ρ, which implies the nullity of $\mu_T 1\!\!1_{t > t_{\gamma(s_0)}}$ near ρ.

The proof is completed by reversing time, which yields $\mathcal{P}(s)$ for $-T^- < t < 0$. $\square$

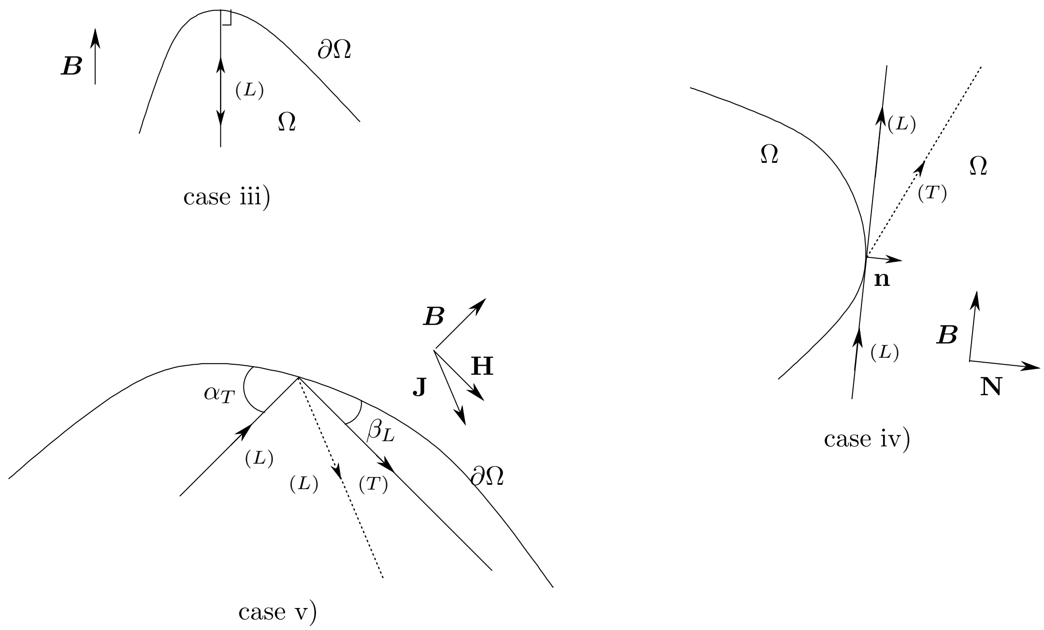

Figure 3. Cases iii), iv) and v)

6. Polynomial decay

We shall prove in this section the following proposition:

Proposition 6.1. *There exist $T, C > 0$ such that for every solution of the Lamé system with initial data:*

$$(u_0, u_1) \in D(\mathcal{L}^N),$$

we have:

$$\|u_0\|^2_{\boldsymbol{H}^1_0} + \|u_1\|^2_{\boldsymbol{L}^2} \leq C \left(\mathcal{Q}^N_T(u) + \|u_0\|^2_{\boldsymbol{L}^2} + \|u_1\|^2_{\boldsymbol{H}^{-1}} \right), \tag{77}$$

where:

- $N = 1$ *if there does not exist on* Ω *any* **boundary** $\boldsymbol{B}$*-resistant ray of infinite life-length;*
- $N = K$ *elsewhere.*

The integer $K \geq 2$ was introduced in theorem 2:

$$K := \sup_{j=1..m} \aleph(\Gamma_j),$$

where Γ_1,...,Γ_m are the spatial images of infinite boundary $\boldsymbol{B}$-resistant rays and $\aleph(\Gamma_j)$ is the minimal order of contact of $\partial\Omega$ with its tangents parallel to $\boldsymbol{B}$ at points of Γ_j. The quadratic form $\mathcal{Q}^T_N$ is defined in section 2 by:

$$\mathcal{Q}^N_T(u) := \sum_{l=0}^{N} \|\mathrm{curl}\,(\partial_t^{l+1} u \wedge \boldsymbol{B})\|^2_{\boldsymbol{H}^{-1}((0,T)\times\Omega)}.$$

Inequality (77) is precisely the sufficient condition of polynomial decay given by proposition 2.7, which completes the proof of theorem 2. As in section 4, we shall argue by contradiction, using the defect measures of section 3.

6.1. **Introduction of measures.** First note that by a density argument, it suffices to show (77) for initial generated by a finite number of eigenfunctions of $\mathcal{L}$. Assume that (77) does not hold. This yields a sequence u^k of smooth solutions of the Lamé system such that:

$$1 = \|u_0^k\|^2_{\boldsymbol{H}_0^1} + \|u_1^k\|^2_{\boldsymbol{L}^2} > k\left(\mathcal{Q}_T^N(u^k) + \|u_0^k\|^2_{\boldsymbol{L}^2} + \|u_1^k\|^2_{\boldsymbol{H}^{-1}}\right) \tag{78}$$

As in section 4, we may assume that u^k converges weakly to 0 in $\boldsymbol{H}^1_{\mathrm{loc}}(\mathbb{R}\times\overline{\Omega})$ and introduce the defect mesures of subsection 3.5. Note that (78) implies:

$$\mathcal{Q}_T^N(u^k) \underset{k\to+\infty}{\longrightarrow} 0. \tag{79}$$

As a consequence, condition (59) of section 4 is fulfilled. In this section, we proved (cf remark 4.3) that this condition implies that in the interval $(0,T)$, the supports of μ_T and μ_L are unions of $\boldsymbol{B}$-resistant rays.

We would like to show, as in section 4, that both measures μ_L and μ_T are null, which would contredict (78). We shall first prove by invariance arguments that, as long as $N \geq 1$, both measures are null in the interior of Ω (subsection 6.2), which yields proposition 6.1 in the favorable case of non-existence of boundary $\boldsymbol{B}$-resistant rays of infinite life length. Subsection 6.3 studies those boundary rays, using traces theorem to restrict (79) to the boundary of Ω.

In all this section, $z = (z_1, z_2, z_3)$ denotes a global spatial orthonormal coordinate system of $\mathbb{R}^3$, in which $\boldsymbol{B} = (B, 0, 0)$. As before, notations y and $x = (t, y)$ shall only be used for local coordinates.

6.2. **Nullity of μ_T and μ_L in the interior of Ω.**

Lemma 6.2. *Let (u^k) be the sequence of solutions of Lamé system introduced in the preceding subsection, with $N \geq 1$. Then:*

- *$\mu_L 1\!\!1_{(0,T)}$ does not depend, in the interior of Ω, on the variables z_2 and z_3;*
- *$\mu_T 1\!\!1_{(0,T)}$ does not depend, in the interior of Ω, on the variable z_1.*

Corollary 6.3. *Under the assumptions of lemma 6.2 and if T is large enough:*

$$\mu_L\big(\{t \in]0, T/2[\}\big) = 0 \tag{80}$$

$$\mu_T\big(\{t \in]0, T/4[,\ z \notin \partial\Omega\}\big) = 0. \tag{81}$$

In particular, if there does not exist any boundary $\boldsymbol{B}$-resistant ray of infinite life-length, (77) holds, with $N = 1$ and large enough T.

Proof of corollary 6.3. The second part of corollary 6.3 is an immediate consequence of (80) and (81).

We shall prove those two conditions by contradiction. Assume that (80) does not hold. Then there is, in the support of μ_L, a point ρ such that:

$$\rho = (t, z, \tau, \zeta), \quad z \in \Omega, \quad t \in]0, T/2[.$$

Consider D, the half-line of origin z with direction ζ. Let z_0 be the first point where D intersects the boundary, $\rho_0 = (t_0, z_0, \tau_0, \zeta_0)$ the point of the longitudinal ray coming from ρ whose spatial projection is z_0, and α the angle between D and the exterior normal vector to the boundary $\mathbf{n}$ in z_0 (see figure 4). If T is large enough (namely if $C_L T/2$ is greater than the diameter of Ω), t_0 is in $(0, T)$. In view of lemma 4.5, whose assumptions are fulfilled because (79) implies (59), one of the three following holds:

- $\alpha = 0$ (corresponding to the $(\mathcal{H}_L)$ case of lemma 4.5);
- $\alpha = \alpha_L = \arctan(c_T/c_L)$ (corresponding to the $(L \to T)$ case of lemma 4.5).
- $\alpha = \frac{\pi}{2}$ (D is tangent to the boundary at z_0, which corresponds to the $(\mathcal{G}_L)$ case of lemma 4.5).

Let $\mathbf{e}$ be an arbitrary vector, orthogonal to $\boldsymbol{B}$. By lemma 6.2, the following points of $S\widehat{Z}_L$ are in the support of μ_L:

$$\rho_\varepsilon := (t, z_\varepsilon, \tau, \zeta), \quad z_\varepsilon := z + \varepsilon\mathbf{e},\ \varepsilon > \mathbf{0} \text{ small.}$$

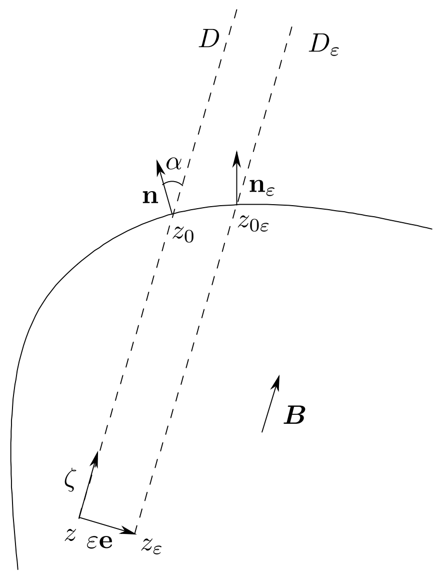

FIGURE 4. Nullity of μ_L.

Consider the half-line D_ε going from z_ε in the direction $\boldsymbol{B}$, and denote by $z_{0\varepsilon}$ its first intersection point with $\partial\Omega$, and by α_ε the angle between n_ε (the exterior normal to the boundary in $z_{0\varepsilon}$), and D_ε. Then:

- if the intersection of D with $\partial\Omega$ in ρ is a transverse intersection (i.e. if $\alpha \neq \frac{\pi}{2}$), then:

$$\rho_{0\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} \rho_0, \quad \alpha_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \alpha,$$

  and the assumption that Ω does not have any contact of infinite order with its tangents implies that for small, non-zero ε, α_ε is close to, but distinct from α, which yields:

$$\alpha_\varepsilon \notin \{0, \alpha_L, \pi/2\}. \tag{82}$$

  As a consequence, ρ_ε cannot be a $\boldsymbol{B}$-admissible point, and D_ε is not the spatial projection of a $\boldsymbol{B}$-resistant ray. For small enough ε, $t_{0\varepsilon}$ is still in the interval $(0,T)$ and thus:

$$\begin{aligned} &\rho_\varepsilon \notin \operatorname{supp}\mu_L, \\ &\rho_\varepsilon \notin \operatorname{supp}\mu_L \text{ (by propagation along the longitudinal flow)}, \\ &\rho \notin \operatorname{supp}\mu_L \text{ (by lemma 6.2)}, \end{aligned}$$

  thus contradicting the definition of ρ;
- if D is tangent at z to $\partial\Omega$, we choose $\mathbf{e} = \mathbf{n}$ (which is orthogonal to $\boldsymbol{B}$). For small ε, the intersection of D_ε and $\partial\Omega$ is a transverse intersection and is close to ρ. Thus, if ε is small but strictly positive, α_ε is close to, but distinct from $\pi/2$, which again shows (82), and as before that ρ is not in the support of μ_L.

Assume now:

$$\mu_T\big(\{t \in]0, T/4[,\ z \in \Omega\}\big) \neq 0.$$

Then there exists a point $\tilde{\rho}$ such that:

$$\tilde{\rho} = (\tilde{z}, \tilde{t}, \tilde{\zeta}, \tilde{\tau}) \in \operatorname{supp}\mu_T,\ \tilde{t} \in]0, T/4[,\ \tilde{z} \in \Omega. \tag{83}$$

Consider a transversal ray passing through $\tilde{\rho}$. It meets the boundary at a non-diffractive point ρ_0, at a certain time $t_0 > \tilde{t}$, after possibly passing through diffractive points. Choose an interior point ρ which is

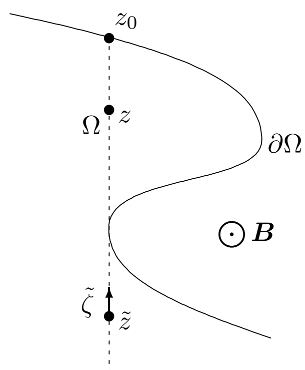

FIGURE 5. Choice of ρ

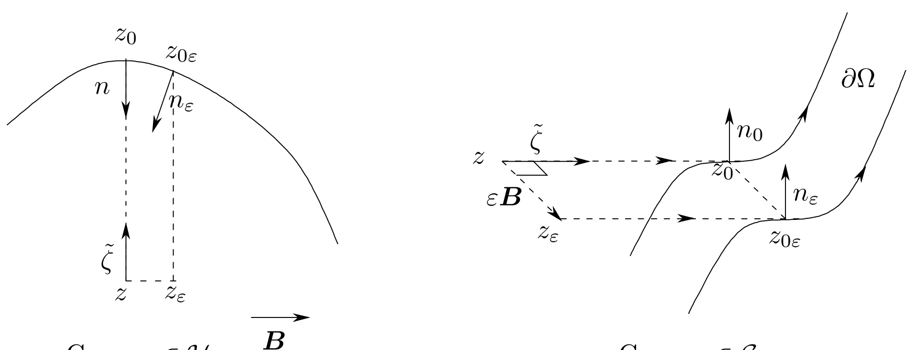

FIGURE 6. Nullity of μ_T in the interior of Ω

located on this transversal ray, after all the diffractive points, but before ρ_0 (see figure 5). Thus:

$$\rho \in \operatorname{supp} \mu_T,\ x \in \Omega,\ t \in]0, T/2[.$$

The condition on t holds for T large enough (if suffices to take T so that the length $c_T T/4$ of a transversal ray of life length $T/4$ is greater than the diameter of Ω).

We may choose coordinates (z_2, z_3) so that $\tilde{\zeta}$ is parallel to $(0,1,0)$. Let $\mathcal{P}$ be the plane passing through z and generated by the two orthogonal vectors $\boldsymbol{B}$ and $\tilde{\zeta}$. Consider $U := \Omega \cap \mathcal{P}$, which is an open subset of $\mathcal{P}$, and the following family of points:

$$\rho_\varepsilon := (t, z_\varepsilon = z + \varepsilon \boldsymbol{B}, \tau, \tilde{\zeta}), \quad |\varepsilon| < \varepsilon_0.$$

For small enough ε_0, z_ε stays in the interior of Ω, so that, in view of lemma 6.2, ρ_ε is in the support of μ_T. Let $\rho_{0\varepsilon}$ be the point of $S_b^*(\mathbb{R} \times \overline{\Omega})$ where the transversal ray coming from ρ hits the boundary. Denote by $\mathbf{n}(\dot{z})$ the exterior unitary normal to $\partial\Omega$ at $\dot{z} \in \partial\Omega$ and, if $\dot{z} \in \mathcal{P}$, by $\mathbf{n}'(\dot{z})$ the exterior unitary normal to ∂U at $\dot{z}$. If $\mathbf{n}(\dot{z})$ is not normal to the vector plane $\mathbf{P}$ generated by $\dot{\zeta}$ and $\boldsymbol{B}$, the (non null) orthogonal projection of $\mathbf{n}(\dot{z})$ on $\mathbf{P}$ is parallel to $\mathbf{n}'(\dot{z})$.

We shall note $\mathbf{n}_0, \mathbf{n}_0', \mathbf{n}_\varepsilon, \mathbf{n}_\varepsilon'$ instead of $\mathbf{n}(z_0), \mathbf{n}'(z_0), \mathbf{n}(z_{0\varepsilon}), \mathbf{n}'(z_{0\varepsilon})$. Two cases arise:

- $\underline{\rho_0 \in \mathcal{H}_T}$. The point ρ_0 is $\boldsymbol{B}$-admissible, and (t_0 being in $(0, T/2)$), μ_L vanishes near ρ_0 by (80), which implies, if μ_T is non null, that ρ_0 is of the type (1) of lemma 4.3. Consequently, $\mathbf{n}_0$ is orthogonal to $\boldsymbol{B}$ but not to $\tilde{\zeta}$ (or else ρ_0 should be in $\mathcal{G}_T$). Thus the (non-null) orthogonal

projection of $\mathbf{n}_0$ on $\mathbf{P}$ is ortogonal to $\boldsymbol{B}$. Hence:

$$\mathbf{n}'_0 \perp \boldsymbol{B}.$$

For small enough ε, $z_{0\varepsilon}$ is close to z_0. Furthermore, the preceding argument is still valid when replacing ρ_0 by $\rho_{0\varepsilon}$, so that $\mathbf{n}'_\varepsilon$ stays orthogonal to $\boldsymbol{B}$. Consequently, ∂U is, in a neighbourhood of z_0, a line segment parallel to $\boldsymbol{B}$, contradicting the assumption that Ω has no contact of infinite order with its tangents.

- $\underline{\rho_0 \in \mathcal{G}_T}$. By the choice of ρ, ρ_0 is not strictly diffractive and is of the $\mathcal{G}_{T^{[2]}}$ type of lemma 4.5, which implies that $\mathbf{n}_0$ is normal to the plane $\mathbf{P}$. On the transversal ray coming from ρ_0, whose spatial projection is a geodesic curve of $\partial\Omega$, $\mathbf{n}$ is still orthogonal to $\boldsymbol{B}$ (at least for time less than T). Consider points $\rho_{0\varepsilon}$, which are close to ρ_0 for small ε. If for one ε such that $|\varepsilon| < \varepsilon_0$, $\rho_{0\varepsilon}$ belongs to $\mathcal{H}_T$, we may reduce to the preceding case with ρ_ε instead of ρ. Thus, we may assume:

$$\forall \varepsilon,\ |\varepsilon| < \varepsilon_0 \Longrightarrow \rho_{0\varepsilon} \in \mathcal{G}_T.$$

The same argument as before shows that along rays coming from $\rho_{0\varepsilon}$, the normal to the boundary $\mathbf{n}$ stays orthogonal to $\boldsymbol{B}$. This yields a small opens subset of $\partial\Omega$ in which $\mathbf{n}$ is orthogonal to $\boldsymbol{B}$, which shows that $\boldsymbol{B}$ is a tangent of infinite order to $\partial\Omega$, contradicting the assumptions on Ω.

□

Proof of lemma 6.2. We may rewrite condition (79):

$$\forall l = 0..N, \quad \partial_t^{l+1} \begin{bmatrix} -\partial_{z_2} u_2^k - \partial_{z_3} u_3^k \\ \partial_{z_1} u_2^k \\ \partial_{z_1} u_3^k \end{bmatrix} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^{-1}((0,T)\times\Omega). \tag{84}$$

When $N \geq 1$, (84) still holds with $l = 1$. The characteristic manifolds of P_T and P_L are disjoint, which shows that (84) holds, in the interior of Ω, if one replaces u^k by u_T^k or u_L^k. Furthermore, on each of these characteristic manifolds, τ does not vanish so that the operator ∂_t^2 is elliptic. Hence:

$$\partial_{z_1} u_T^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\mathrm{loc}}((0,T)\times\overline{\Omega}) \tag{85}$$

$$\partial_{z_2} u_{L1}^k,\ \partial_{z_3} u_{L1}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^1_{\mathrm{loc}}((0,T)\times\overline{\Omega}). \tag{86}$$

We have used the nullity of $\operatorname{div} u_T^k$ to get (85), and the nullity of $\operatorname{curl} u_L^k$ on the two last lines of (84) to get (86). We shall use properties (85), (86) of convergence to 0 with gain of one derivative in a classical way (see the proof of the propagation of defect measures), calculating the commutator of the appropriate operators (∂_{z_1} for u_T^k, ∂_{z_2} and ∂_{z_3} for u_L^k) with a "test" pseudo-differential operator of order 2. For example, (85) implies, by integration by parts:

$$\forall A \in \mathcal{A}_i^2,\ ([A, \partial_{z_1}] u_T^k, u_T^k) \underset{k\to+\infty}{\longrightarrow} 0$$

$$\left\langle \mu_T, \partial_{z_1} \frac{a_2}{2\nu_T^2 \tau^2} \right\rangle = 0.$$

(We used that on the support of μ_T, $\nu_T^2 \tau^2 = \|\eta\|^2$.) This shows that μ_T does not depend upon z_1. To prove that μ_L does not depend upon z_2 nor z_3, it suffices to prove the same property for the defect measure μ_{L1} of u_{L1}^k (μ is polarized along $\boldsymbol{B}$). To do so, we use the above argument on (86). □

6.3. **Nullity of μ_T on the boundary of Ω.** We now assume that there exists a boundary $\boldsymbol{B}$-resistant ray, γ, of infinite life length. It is transversal and its spatial projection lives on a plane curve Γ, contained in the intersection of $\partial\Omega$ with a plane $\mathcal{P}$ normal to $\boldsymbol{B}$. Furthermore, all points of γ being gliding points, Γ is the boundary of a convex set of $\mathcal{P}$. We choose T large enough so that in the interval $(0,T)$, the spatial image of γ is the entire curve Γ. We shall work near a point $\tilde{z}$ of Γ, such that $\boldsymbol{B}$ is tangent at the order $\aleph_0 = \aleph(\Gamma)$ at $\tilde{z}$ to $\partial\Omega$. This choice is made possible by the definition of $\aleph_0$. Recall that by assumption, $N \geq \aleph_0$. Let $\tilde{\rho}$ be a point in the image of γ whose spatial projection is $\tilde{z}$.

Notations. We choose $\tilde{z}$ as the origin of the orthonormal frame $(0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and assume $\mathbf{e}_2$ to be tangent, in 0, to Γ and $\mathbf{e}_3$ equal to the exterior unitary normal at 0 to $\partial\Omega$. Consider the local coordinates (s_1, s_2) on $\partial\Omega$ defined by:

$$s_1 = z_1,\ s_2 = z_2.$$

The assumption on $\tilde{z}$ implies that near 0:

$$z_3(s_1,s_2) = s_1^{\aleph_0} T_0(s_1,s_2), \quad \frac{\partial z_3}{\partial_{s_1}}(s_1,s_2) = s_1^{\aleph_0 - 1} T_1(s_1,s_2) \quad T_j(0,0) \neq 0. \tag{87}$$

The strategy of this last part of the proof is a simple one. We shall restrict condition (79) to the boundary of Ω by an appropriate trace theorem. Such a theorem does not exist, in general, in spaces H^s, $s \leq 1/2$, but in our particular case, u_T^k and u_L^k being solutions of a differential equation which is transverse to the boundary, it is possible to take the trace of (79) on the boundary, losing as in standard trace theorems only half a derivative (cf proposition 7.2 in the appendix). The boundary equations thus obtained will yield the nullity of μ_T, by a simple lemma giving bounds of L^2 norms with loss of derivatives (lemma 6.4), and the usual boundary equations given by standard hyperbolic, elliptic and glancing theory.

First step: restriction to the boundary. We shall first prove the following:

$$\operatorname{curl}(u_T^k \wedge \boldsymbol{B})_{\restriction\partial\Omega} = B \begin{pmatrix} -\partial_{z_2} u_{T2}^k - \partial_{z_3} u_{T3}^k \\ \partial_{z_1} u_{T2}^k \\ \partial_{z_1} u_{T3}^k \end{pmatrix}_{\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\text{loc}}^{N-1/2}((0,T)\times\partial\Omega) \tag{88}$$

$$\operatorname{curl}(u_L^k \wedge \boldsymbol{B})_{\restriction\partial\Omega} = B \begin{pmatrix} -\partial_{z_2} u_{L2}^k - \partial_{z_3} u_{L3}^k \\ \partial_{z_1} u_{L2}^k \\ \partial_{z_1} u_{L3}^k \end{pmatrix}_{\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\text{loc}}^{N-1/2}((0,T)\times\partial\Omega). \tag{89}$$

First of all we need to decouple condition (79) into one condition on the longitudinal wave and one condition on the transversal wave. Set:

$$w^k := \operatorname{curl}(\partial_t^{N+1} u^k \wedge \boldsymbol{B}) = w_T^k + w_L^k$$
$$w_{T,L}^k := \operatorname{curl}(\partial_t^{N+1} u_{T,L}^k \wedge \boldsymbol{B}).$$

Then:

$$\nu_T^2 \partial_t^2 w_T^k - \Delta w_T^k = 0, \quad \nu_L^2 \partial_t^2 w_L^k - \Delta w_L^k = 0$$

Adding these two equations and using (79) we get:

$$\nu_T^2 \partial_t^2 w_T^k + \nu_L^2 \partial_t^2 w_L^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^{-3}((0,T)\times\Omega). \tag{90}$$

Furthermore, condition (79) twice differentiated with respect to time yields:

$$\partial_t^2 w_T^k + \partial_t^2 w_L^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^{-3}((0,T)\times\Omega). \tag{91}$$

We deduce from (90) and (91), ν_T and ν_L being distincts:

$$\partial_t^2 w_{T,L}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^{-3}((0,T)\times\overline{\Omega}).$$

Both functions w_T^k and w_L^k being solutions of wave equations, proposition 7.2 of the appendix implies:

$$\operatorname{curl}\left(\partial_t^{N+3} u_{T,L}^k \wedge \boldsymbol{B}\right)_{\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\text{loc}}^{-7/2}((0,T)\times\partial\Omega)$$

Thus, if $\tilde{\rho} = (\tilde{t}, \tilde{y}', \tilde{\tau}, \tilde{\eta}')$ is a boundary point, and if $\tilde{\tau}$ is non null, the ellipticity of ∂_t^{N+3} yields:

$$\operatorname{curl}\left(u_{T,L}^k \wedge \boldsymbol{B}\right)_{\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}_{\tilde{\rho},\partial}^{N-1/2}.$$

It remains to prove the same property when $\tilde{\tau}=0$. In this case $\tilde{\rho}\in\mathcal{E}_T\cap\mathcal{E}_L$. Let $M>0$. The standard elliptic theory (propositions 3.18 for the first line and 3.17 for the following) implies:

$$\partial_{x_n} u^k_{\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^M_{\tilde{\rho},\partial} \tag{92}$$

$$D_{x_n} u^k_{T\restriction x_n=0} + \Xi_T u^k_{T\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0, \text{ in } \boldsymbol{H}^M_{\tilde{\rho},\partial} \tag{93}$$

$$D_{x_n} u^k_{L\restriction x_n=0} + \Xi_L u^k_{L\restriction x_n=0} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^M_{\tilde{\rho},\partial} \tag{94}$$

$$\Xi_{T,L}\in\mathcal{A}^1,\quad \sigma_1(\Xi_{T,L}) = i\sqrt{\|\eta'\|^2-\nu_{T,L}\tau^2} \text{ near } \tilde{\rho}.$$

Adding (93) and (94), and then using (92), we get:

$$\Xi_T u^k_T + \Xi_L u^k_L \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^M_{\tilde{\rho},\partial}.$$

This shows, using the Dirichlet boundary condition on u^k and the ellipticity of the operator $\Xi_T-\Xi_L$ that:

$$u^k_{T,L} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{H}^M_{\tilde{\rho},\partial}.$$

In view of (93) and (94), the same property holds in the space $\boldsymbol{H}^{M-1}$ on $\partial_{x_n}u^k_{T,L}$, which completes the proof of (88) and (89).

Second step. We shall now prove that under the assumptions of subsection 6.1, μ_T is polarized along $\boldsymbol{B}$:

$$\pi_B\mu_T\pi_B \mathbb{1}_{t\in(0,T)} = \mu_T \mathbb{1}_{t\in(0,T)}.$$

In other terms, denoting by u_j the jth component of u in the basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$:

$$u^k_{T2} \underset{k\to+\infty}{\longrightarrow} 0,\quad u^k_{T3} \underset{k\to+\infty}{\longrightarrow} 0.$$

First of all, we prove the following:

$$\partial_n u^k_{j\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } L^2_{\tilde{\rho},\partial},\ j=2,3. \tag{95}$$

The sum of the last line of relations (88) and (89) yields:

$$\partial_{z_1} u^k_2 \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H^{N-1/2}_{\text{loc}}\left((0,T)\times\partial\Omega\right) \tag{96}$$

On the other hand:

$$\partial_{s_1} u^k_{2\restriction\partial\Omega} = \partial_{z_1} u^k_{2\restriction\partial\Omega} + \frac{\partial z_3}{\partial s_1}\partial_{z_3} u^k_{2\restriction\partial\Omega}.$$

The Dirichlet boundary condition on u^k allows us to take out, in the preceding equality, all tangential derivatives. In view of (87), and noting the $\frac{\partial}{\partial n}$-component of $\frac{\partial}{\partial z_3}$ does not vanish at 0, we deduce from (96) that in a neighbourhood U_0 of 0 in $\partial\Omega$:

$$s_1^{\aleph_0-1}\partial_n u^k_{2\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H^{N-1/2}_{\text{loc}}(U_0). \tag{97}$$

Lemma 6.4 below allows us to take out the factor $s_1^{\aleph_0-1}$ in 97, in return for the loss of $\aleph_0-1$ derivatives:

Lemma 6.4. *Let $r>-1/2$, $p>0$, $d\in\mathbb{N}^*$ and $f\in H^{r+p}(\mathbb{R}^d)$, with compact support. Let $y=(y_1,..,y_d)$ be the canonical coordinates on $\mathbb{R}^d$. Then:*

$$\|f\|_{H^r}\leq C\|y_1^p f\|_{H^{r+p}}.$$

Proof. It is sufficient to prove the inequality with $p=1$ and $f\in C_0^\infty(\mathbb{R}^d)$. Set:

$$N(f):=-\mathrm{Re}\int(1+|\eta|^2)^r\eta_1\frac{\partial\hat{f}}{\partial\eta_1}\bar{\hat{f}}\,d\eta$$

Cauchy-Schwarz inequality implies:

$$|N(f)|\leq\|f\|_{H^r}\|y_1f\|_{H^{r+1}}, \tag{98}$$

A simple integration by parts yields:

$$\begin{aligned}
N(f)=&-\frac{1}{2}\int\frac{\partial}{\partial_{\eta_1}}|\hat{f}|^2(1+|\eta|^2)^r\eta_1\,d\eta\\
=&\int|\hat{f}|^2(1+|\eta|^2)^{r-1}\left\{\frac{1}{2}+\frac{1}{2}|\eta|^2+r\eta_1^2\right\}d\eta\\
N(f)\geq& c_r\|f\|^2_{H^r},\quad c_r>0. \qquad (99)
\end{aligned}$$

To get inequality (99), we used the assumption $r+1/2>0$. Inequalities (98) and (99) yield the announced result. □

From (97), lemma 6.4 and the assumption $N-\aleph_0\geq 0$ we get (95) with $j=2$. A similar argument yields the same result on $\partial_n u_3^j$.

If $\tilde{\rho}\in\mathcal{H}_L$ (i.e. when $c_T>c_L$) we have:

$$u^k_{L\restriction\partial\Omega}\longrightarrow 0 \text{ in } \boldsymbol{H}^1_{\tilde{\rho},\partial\Omega}$$

which implies by boundary Dirichlet condition on u^k the same property on u^k_T.

In the elliptic case, standard elliptic theory (lemma 3.17) yields the following boundary condition:

$$\forall j\in\{2,3\},\ D_nu^k_{Lj}=\Xi_1u^k_{Lj}+o(1)\text{ in }L^2_{\tilde{\rho}}.$$

This condition is still valid with T instead of L, by (95) and the Dirichlet boundary condition on u^k.

Let μ'_T be the defect measure of the sequence (u^k_{T2},u^k_{T3}). Denoting by π_{23} the orthogonal projection of $\mathbb{C}^3$ on the plane $(\mathbf{e}_2,\mathbf{e}_3)$, one may identify μ'_T with $\pi_{23}\mu_T\pi_{23}$. The support of μ'_T is, in a neighbourhood of $\tilde{\rho}$, contained in $\mathcal{G}^T$ and its spatial projection is contained in Γ. According to the propagation theorem of N. Burq and G. Lebeau [2, theorem 1], there exists a function M, continuous (except possibly at hyperbolic points) and inversible on the support of μ'_T, such that $M^*\mu'_TM$ propagates along the transversal flow Φ_T. The **scalar** boundary conditions above show that one may also apply the propagation theorem on each component u^k_{T2} and u^k_{T3}, so that M is of the form $m\mathrm{Id}_{\mathbb{C}^2}$, where m is a complex valued function. Furthermore, μ_T is polarized orthogonally to the direction of propagation, which is, along γ, the direction tangential to Γ. As a consequence, denoting by π_ρ the orthogonal projection of $\mathbb{C}^2$ on the line generated by (ζ_2,ζ_3), the following equality yields near Γ:

$$\pi_\rho\mu'_T=\mu'_T,\quad t\in(0,T)$$

by scalar propagation, we get, for $t\in(0,T)$:

$$\forall s,\ \pi_{\Phi_T(s,\rho)}\mu'_T=\mu'_T.$$

This is impossible unless $\mu'_T=0$. Hence:

$$u^k_{Tj}\longrightarrow 0\text{ in }H^1_{\tilde{x}}((0,T)\times\Omega),\quad i\in\{2,3\}.$$

Third step. We are now able to conclude to the nullity of μ_T. The two first lines of (89) may be rewritten, using the nullity of $\operatorname{curl} u_L^k$:

$$(\partial_{z_2} u_{L1}^k, \partial_{z_3} u_{L1}^k) \to 0 \text{ in } H_{loc}^{N-1/2}((0,T)\times\partial\Omega).$$

Noting that $\partial_{s_2} u = \partial_{z_2} u + \frac{\partial z_3}{\partial_{s_2}} \partial_{z_3} u$, we get:

$$\partial_{s_2} u_{T1}^k = -\partial_{s_2} u_{1L}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\mathrm{loc}}^{N-1/2}((0,T)\times\partial\Omega), \tag{100}$$

which yields:

$$u_{T1\restriction x_n=0}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\tilde\rho,\partial}^1 \tag{101}$$

Consider now the first line of (88), :

$$\partial_{z_1} u_{T1\restriction\partial\Omega}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{loc}^{N-1/2}((0,T)\times\partial\Omega). \tag{102}$$

We have:

$$\frac{\partial}{\partial s_2}\left(\partial_{z_1} u_{T1\restriction\partial\Omega}^k - \frac{\partial z_3}{\partial s_1}\partial_{z_3} u_{T1\restriction\partial\Omega}^k\right) = \partial_{s_1}\partial_{s_2} u_{T1\restriction\partial\Omega}^k.$$

Together with (100) and (102) this yields:

$$\frac{\partial}{\partial s_2}\left(\frac{\partial z_3}{\partial s_1}\partial_{z_3} u_{T1\restriction\partial\Omega}^k\right) \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\mathrm{loc}}^{N-3/2}((0,T)\times\partial\Omega).$$

As a consequence, in view of the expression (87) of z_3, there exists a neighbourhood U_0 of 0 in $\partial\Omega$ such that:

$$s_1^{\aleph_0-1}\frac{\partial}{\partial s_2}\left(T_1(s_1,s_2)\partial_n u_{T1\restriction\partial\Omega}^k\right) \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\mathrm{loc}}^{N-3/2}((0,T)\times U_0).$$

With lemma 6.4 one gets:

$$\frac{\partial}{\partial s_2}\left(g(s_1,s_2)\partial_n u_{T1\restriction\partial\Omega}^k\right) \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\mathrm{loc}}^{N-\aleph_0-1/2}((0,T)\times U_0),$$

which yields, using the ellipticity of the operator ∂_{s_2} at $\tilde\rho$:

$$\partial_n u_{T1\restriction\partial\Omega} \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } H_{\tilde\rho,\partial}^{N-\aleph_0+1/2}.$$

Thus, since $N \geq \aleph_0$:

$$\partial_n u_{T1\restriction\partial\Omega}^k \underset{k\to+\infty}{\longrightarrow} 0 \text{ in } \boldsymbol{L}_{\tilde\rho,\partial}^2. \tag{103}$$

Conditions (101) and (103) imply, by point b) of lemma 3.19, the nullity of μ_{T1}, component of μ_T along $\boldsymbol{B}$. This complete the proof of the nullity of μ_T in view of our second step above.

7. Appendix

7.1. **Two useful results on boundary value problems.** We shall state here two lemmas concerning solutions of a partial diffential equation which is transverse to the boundary of an open set. The first one says that for such functions, the control of derivatives which are tangential to the boundary suffices to control all the derivatives. The second one states a trace theorem, with loss of one-half derivative, in all H^s spaces, even if $s \leq 1/2$. As in section 3, we shall work in an open subset $\overline{X}$ of $\overline{\mathbb{R}}_+^{n+1}$, of the form $X' \times [0,l[$, where X' is an open subset of $\mathbb{R}^n$. Let P be a diffential operator of degree r on $\overline{X}$, of the following form:

$$P = \sum_{j=0..r} Q_{r-j}\partial_{x_n}^j,$$

where the Q_j's are $N\times N$ matrices of tangential differential operators, with $C^\infty\left(\overline{X}\right)$ coefficients, and Q_0 is the identity of $\mathbb{C}^N$. To simplify the following statements, we suppose $u \in C^\infty(\overline{X})$.

Proposition 7.1. *Let $s \geq 0$, $j \in \mathbb{N}$, and suppose that $Pu = 0$ on $\overline{X}$. Then:*

$$\forall \varphi \in C_0^\infty\left(\overline{X}\right), \quad \exists \tilde{\varphi} \in C_0^\infty\left(\overline{X}\right), \quad \|\varphi \partial_{x_n}^j u\|_{\boldsymbol{H}^{s-j}} \leq C\|\tilde{\varphi} u\|_{L^2(0,l;\boldsymbol{H}^s(X'))},$$

where C does not depend on u. Likewise, if again $Pu = 0$ then:

$$\forall \varphi \in C_0^\infty\left(\overline{X}\right), \quad \exists \tilde{\varphi} \in C_0^\infty\left(\overline{X}\right), \quad \|\varphi u\|_{L^2(0,l;\boldsymbol{H}^{-s}(X'))} \leq C\|\tilde{\varphi} u\|_{\boldsymbol{H}^{-s}(X)}.$$

Proposition 7.2. *Let $s \in \mathbb{R}$, $j \in \mathbb{N}$ and suppose that $Pu = 0$ on $\overline{X}$. Then:*

$$\forall \varphi \in C_0^\infty\left(\overline{X}\right), \quad \exists \tilde{\varphi} \in C_0^\infty\left(\overline{X}\right), \quad \|\varphi \partial_{x_n}^j u_{\restriction x_n=0}\|_{\boldsymbol{H}_{\text{loc}}^{s-j-1/2}} \leq C\|\tilde{\varphi} u\|_{\boldsymbol{H}^s(X)},$$

where C does not depend on u.

7.2. **Proof of proposition 3.19.** As mentionned in the introduction of section 3, whe shall assume that each u^k is smooth enough, so that all the quantities appearing in the following calculation are well defined and finite. The general result may be obtained with a technical smoothing argument (cf [2, lemma 2.8]). Let:

$$A = A_0 D_{x_n},\ A_0 \in \mathcal{A}^0,\ \mathcal{C}^k := \left([P, A]u^k, u^k\right).$$

Take the support of A_0 in a small enough neighbourhood of $\tilde{\rho}$. The operator P is formally self-adjoint. A simple integration by parts yields:

$$\mathcal{C}^k = \underbrace{-(APu^k, u^k) + (Au^k, Pu^k)}_{0} - (Au^k_{\restriction x_n=0}, iD_{x_n}u^k_{\restriction x_n=0})_\partial + (iD_{x_n}Au^k_{\restriction x_n=0}, u^k_{\restriction x_n=0})_\partial,$$

where $(.,.)_\partial$ is the $\boldsymbol{L}^2$ scalar product on $\{x_n = 0\}$, with respect to the measure $\sqrt{\mathrm{g}_{\restriction x_n=0}}dx'dt$. We have:

$$\begin{aligned} D_{x_n}A_0D_{x_n}u^k =& A_0D_{x_n}^2u^k + [D_{x_n}, A_0]D_{x_n}u^k \\ =& -A_0Qu^k + R_0D_{x_n}u^k, \quad R_0 \in \mathcal{A}^0 \\ \mathcal{C}^k =& i(A_0D_{x_n}u^k, D_{x_n}u^k)_\partial - i(A_0Qu^k, u^k)_\partial + (R_0D_{x_n}u^k, u^k)_\partial. \end{aligned}$$

So, using condition (44),

$$\begin{aligned} \mathcal{C}^k = i(A_0D_{x_n}u^k, D_{x_n}u^k)_\partial - i(A_0QB_{-1}D_{x_n}u^k, B_{-1}D_{x_n}u^k)_\partial - i(A_0Qh^k, B_{-1}D_{x_n}u^k)_\partial \\ - i(A_0QB_{-1}D_{x_n}u^k, h^k)_\partial + (R_0D_{x_n}u^k, B_{-1}D_{x_n}u^k)_\partial + o(1),\ k \to +\infty \end{aligned}$$

Take A_0 of the form $T_0^*T_0$, where $T_0 \in \mathcal{A}_0$ is scalar, elliptic at $\tilde{\rho}$ and has support in a small neighbourhood of $\tilde{\rho}$. Then:

$$\mathcal{C}^k = i(E_0D_{x_n}u^k, D_{x_n}u^k)_\partial - i(T_0Qh^k, B_{-1}T_0D_{x_n}u^k) - i(QB_{-1}T_0D_{x_n}u^k, T_0h^k) + o(1),\ k \to +\infty$$

$$E_0 \in \mathcal{A}^0,\ E_0 = T_0^*T_0 - B_{-1}^*T_0^*T_0QB_{-1} + B_{-1}^*R_0.$$

Denoting by t_0 the principal symbol (which is scalar) of T_0, we have:

$$\sigma_0(E_0) = |t_0|^2\left(1 - {}^t b_{-1}^* b_{-1} q_2\right).$$

Since q_2 vanishes at $\tilde{\rho}$, we may choose t_0 with support in a small enough neighbourhood of $\tilde{\rho}$ such that:

$$\sigma(E_0) \geq 1/2|t_0|^2$$

(in the sens of quadratic positive hermitian forms). The weak Gårding inequality, applied to the operator $E_0 - 1/2T_0^*T_0$, thus yields:

$$\liminf_{k\to+\infty} \operatorname{Re}\left(E_0D_{x_n}u^k, D_{x_n}u^k\right) - \frac{1}{2}\|T_0D_{x_n}u^k\|^2_{\boldsymbol{L}^2_\partial} \geq 0.$$

This implies, using the convergence to 0 of h^k in $\boldsymbol{H}^1_{\tilde{\rho}}$:

$$\liminf_{k\to+\infty} \operatorname{Im}\mathcal{C}^k \geq \frac{1}{4}\|T_0D_{x_n}u^k\|^2_{\boldsymbol{L}^2_\partial}.$$

Thus $D_{x_n} u^k_{\restriction x_n=0}$ is bounded in $\boldsymbol{L}^2_{\tilde\rho,\partial}$, which yields, with the boundary condition, that $u^k_{\restriction x_n=0}$ is bounded in $\boldsymbol{H}^1_{\tilde\rho,\partial}$. The proof of (49) is complete.

When μ is null near $\tilde\rho$, we have:

$$\lim_{k\to+\infty} \mathcal{C}^k = \big\langle \mu, \frac{\{p, a_1\xi_n\}}{\tau^2} \big\rangle = 0,$$

which yields (50).

Point b) of proposition 3.19 may be seen as a consequence of the propagation theorem of [2]. The assumptions (51) imply that **any** uniform Lopatinsky boundary condition holds on the traces of u^k, which shows that the measure μ propagates near $\tilde\rho$ with any smooth multiplicative factor, which is impossible unless μ is null.

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